Soliton approach to the noisy Burgers equation: Steepest descent method

Hans C. Fogedby

Institute of Physics and Astronomy, University of Aarhus, DK-8000, Aarhus C, Denmark*
and NORDITA, Blegdamsvej 17, DK-2100, Copenhagen Ø, Denmark

The noisy Burgers equation in one spatial dimension is analyzed by means of the Martin-Siggia-Rose technique in functional form. In a canonical formulation the morphology and scaling behavior are accessed by means of a principle of least action in the asymptotic nonperturbative weak noise limit. The ensuing coupled saddle point field equations for the local slope and noise fields, replacing the noisy Burgers equation, are solved yielding nonlinear localized soliton solutions and extended linear diffusive mode solutions, describing the morphology of a growing interface. The canonical formalism and the principle of least action also associate momentum, energy, and action with a soliton-diffusive mode configuration and thus provide a selection criterion for the noise-induced fluctuations. In a ''quantum mechanical'' representation of the path integral the noise fluctuations, corresponding to different paths in the path integral, are interpreted as ''quantum fluctuations'' and the growth morphology represented by a Landau-type quasiparticle gas of ''quantum solitons'' with gapless dispersion $E \propto P^{3/2}$ and ''quantum diffusive modes'' with a gap in the spectrum. Finally, the scaling properties are discussed from a heuristic point of view in terms of a ''quantum spectral representation'' for the slope correlations. The dynamic exponent $z = 3/2$ is given by the gapless soliton dispersion law, whereas the roughness exponent $\zeta = 1/2$ follows from a regularity property of the form factor in the spectral representation. A heuristic expression for the scaling function is given by a spectral representation and has a form similar to the probability distribution for Lévy flights with index z. [S1063-651X(98)09604-4]

I. INTRODUCTION

This is the second of a series of papers where we analyze the Burgers equation in one spatial dimension with the purpose of modeling the growth of an interface; for a brief account we refer to [1]. In the first paper, denoted in the following by I [2], we investigated the *noiseless* Burgers equation [3–6] in terms of its nonlinear soliton or shock wave excitations and linear diffusive modes. In the present paper we address our main objective, namely the *noisy* Burgers equation in one spatial dimension [7]. This equation has the form

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \lambda u \nabla u + \nabla \eta, \qquad (1.1)$$

where ν is a damping constant or viscosity and λ a nonlinear coupling strength. The equation is driven by a conserved white noise term, $\nabla \eta$, where η has a Gaussian distribution and is short-range correlated in space according to

$$\langle \eta(x,t) \eta(x',t') \rangle = \Delta \delta(x-x') \delta(t-t'). \qquad (1.2)$$

In the context of modeling a growing interface the Kardar-Parisi-Zhang equation (KPZ) [8] for the height field h,

$$\frac{\partial h}{\partial t} = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta, \qquad (1.3)$$

is equivalent to the Burgers equation by means of the relationships

$$u = \nabla h, \qquad (1.4)$$

$$h = \int u \, dx, \qquad (1.5)$$

that is, the Burgers equation governs the dynamics of the local slope of the interface. In Fig. 1 we have sketched the growth morphology in terms of the height and slope fields.

The substantial conceptual problems encountered in non-equilibrium physics are in many ways embodied in the Burgers-KPZ equations (1.1) and (1.3), which describe the self-affine growth of an interface subject to annealed noise arising from fluctuations in the drive or in the environment [9–15]. Interestingly, the Burgers-KPZ equations are also encountered in a variety of other problems such as randomly stirred fluids [7], dissipative transport in a driven lattice gas [16–18], the propagation of flame fronts [19–21], the sine-Gordon equation [22], and magnetic flux lines in superconductors [23]. Furthermore, by means of the Cole-Hopf transformation [24,25] the Burgers-KPZ equations are also related to the problem of a directed polymer [26,27] or a quantum particle in a random medium [28,29] and thus to the theory of spin glasses [30–32].

In contrast to the case of the noiseless Burgers equation discussed in paper I where the slope field eventually relaxes due to the dissipative term $\nu \nabla^2 u$, unless energy is supplied to the system at the boundaries, the noisy Burgers equation (1.1) describes an open nonlinear dissipative system driven into a stationary state with random energy input at a short wavelength scale provided by the conserved noise $\nabla \eta$. In the stationary regime the equation thus describes time-independent stochastic self-affine roughened growth. In the
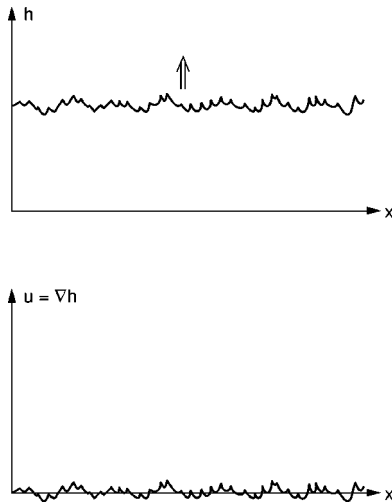
FIG. 1. We depict the general growth morphology for a 1D interface in terms of the slope field u and the height field h (arbitrary units).

linear case for $\lambda = 0$ the Burgers equation reduces to the noise-driven Edwards-Wilkinson equation (EW) [33]

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \nabla \eta; \qquad (1.6)$$

here for the slope field u. Owing to the absence of the nonlinear growth term $\lambda u \nabla u$ the cascade in wave number space is absent and the correlations, probability distributions, and scaling properties are easy to derive [34]. Furthermore, since the EW equation is compatible with a fluctuation-dissipation theorem, it actually describes the dynamic fluctuations in an equilibrium state with temperature $\Delta/2\nu$ (in units such that $k_B = 1$) and as a consequence does not provide a proper description of a growing interface. On the other hand, the presence of the nonlinear growth term $\lambda u \nabla u$ in Eq. (1.1) renders it much more complicated and much richer. The term filters the input noise $\nabla \eta$ and gives rise to interactions between different wave number components leading to a cascade that changes both the scaling properties and the probability distributions from the linear EW case.

The Burgers-KPZ equations owing to their simple form accompanied by their very complex behavior have served as paradigms in the theory of driven and disordered systems and have been studied intensively [9–15]. One set of issues that has have been addressed is the scaling properties [35–38]. According to the dynamic scaling hypothesis [35,38] supported by numerical simulations and the fixed point structure of a renormalization group scaling analysis, [7,8], the slope field u is statistically scale invariant in the sense that the self-affine rescaled $u'(x,t) = b^{-(\zeta-1)} u(bx, b^z t)$ is statistically equivalent to $u(x,t)$, where b is a scale parameter. More precisely, the scaling hypothesis implies the following dynamical scaling form for the slope correlation function in the stationary regime [7,8,38–40]:

$$\langle u(x,t) u(x,'t') \rangle = |x-x'|^{2(\zeta-1)} f(|t-t'|/|x-x'|^z). \qquad (1.7)$$

The scaling behavior in the long wavelength–low frequency limit is thus governed by two scaling dimensions: (i) the

TABLE I. Exponents and universality classes.

Universality class	ζ	z
EW	1/2	2
Burgers-KPZ	1/2	3/2

roughness or wandering exponent ζ, characterizing the slope correlations for a stationary profile and (ii) the dynamic exponent z, describing the temporal scaling in the stationary regime [41]. The slope field u has the scaling dimension $1 - \zeta$. For large w the scaling function $f(w) \propto w^{-2(1-\zeta)/z}$; for small w $f(x) \propto$ const.

Two properties determine the scaling exponents, namely, a scaling law and an effective fluctuation-dissipation theorem. Like the noiseless or deterministic Burgers equation discussed in paper I, the noisy equation is also invariant under a Galilean transformation [7,8]

$$x \rightarrow x - \lambda u_0 t, \qquad (1.8)$$

$$u \rightarrow u + u_0. \qquad (1.9)$$

Since the nonlinear coupling strength λ here enters as a structural constant of the Galilean symmetry group it transforms trivially under a scaling transformation and combined with the existence of a nonvanishing fixed point for the effective coupling strength $\lambda^2 \Delta/\nu^3$ we infer the scaling law [7–10]

$$\zeta + z = 2, \qquad (1.10)$$

relating ζ and z. Furthermore, noting that the stationary Fokker-Planck equation for the Burgers equation (1.1) is solved by a Gaussian distribution [7,11,27]

$$P(u) \propto \exp\left[-\frac{\nu}{\Delta} \int dx u^2 \right], \qquad (1.11)$$

independent of λ it follows that u is an independent random variable and that the height variable h according to Eq. (1.5) performs random walk, corresponding to the roughness exponent $\zeta = 1/2$, also in the linear EW case. From the scaling law (1.10) we subsequently obtain the dynamic exponent $z = 3/2$. In the linear EW case $\zeta = 1/2$ and $z = 2$, characteristic of diffusion. In Table I we have summarized the exponents for the EW and Burgers-KPZ universality classes.

The standard tool used in the analysis of the scaling properties of nonlinear Langevin equations of the type in Eqs. (1.1)–(1.3) is the dynamic renormalization group (DRG) method [35]. This approach is based on an expansion in powers of the nonlinear couplings and a subsequent term by term average over the noise, implementing the statistical average. Power counting or the degree of divergence of the perturbative corrections typically identifies a critical dimension separating regions where infrared convergent perturbation theory holds yielding mean field behavior from regions with infrared divergent expansions. In the divergent regions the expansion is regularized by (i) a momentum shell integration in the short wavelength limit, corresponding to a wave number version of the Kadanoff construction [35], or (ii) more powerful field theoretical dimensional regulariza-

tion schemes [42]. The DRG thus allows for an organization of the divergent terms and leads to renormalization group equations for the effective parameters in the theory, often in terms of an epsilon expansion about the critical dimension, yielding expressions for the critical exponents and information about the scaling functions.

For the Burgers-KPZ equations the scaling phenomenology is now well understood [7,8,43,44]. The expansion in powers of the nonlinear coupling strength λ in conjunction with a term by term noise average and an appropriate regularization scheme, e.g., a momentum shell integration, yields DRG equations for the scaling behavior in terms of an effective coupling strength $g = \lambda^2 \Delta / \nu^3$. The lower critical dimension is $d_c = 2$, also following from simple power counting. Below $d_c = 2$ there appear three renormalization group fixed points: An infrared-unstable Gaussian fixed point $g^* = 0$, corresponding to vanishing coupling strength, describing a smooth interface governed by the EW equation, an infrared-stable strong coupling fixed point $g^* \neq 0$, characterizing a rough interface, and an infrared-unstable fixed point for $g^* = \infty$. The exponents assume nontrivial values for all $\lambda \neq 0$. In $d = 1$ an effective fluctuation-dissipation theorem equivalent to the Gaussian form in Eq. (1.11) is operative, and together with the Galilean invariance, implying trivial scaling of λ, the renormalization group equations yield the exponents in Table I associated with the infrared-stable nontrivial strong coupling fixed point [7,8,43,44]. Above $d_c = 2$ the role of the fixed points is reversed and we have an infrared stable vanishing Gaussian fixed point $g^* = 0$, corresponding to a smooth interface described by the EW equation and a nontrivial infrared-unstable fixed point $g^* \neq 0$, which can be determined by an epsilon expansion about $d_c = 2$ just as in the case of the nonlinear σ model [42]. This DRG phenomenology indicates the existence of a kinetic phase transition from a weak coupling smooth phase for small g to a strong coupling rough phase that does not seem accessible by analytical methods [43–45]; however, the scaling exponents in the strong coupling phase has been determined numerically [9,46] and by self-consistent mode coupling theory [47–49]. The issue of an upper critical dimension has also been considered and, for example, proposed to be $d = 4$ in Ref. [50]; the issue, however, remains controversial and we refer to [50] for further referencing.

Although the scaling exponents associated with the strong coupling fixed point in $d = 1$ are exactly known, owing to Galilean invariance and the fluctuation-dissipation theorem, and confirmed to second loop order [43,44], the scaling function does not follow from a simple DRG analysis. In fact, unlike the driven lattice gas, which can be analyzed by an epsilon expansion below $d = 2$, this is not the case for the Burgers-KPZ equations, which are related to the continuum limit of the lattice gas. On the other hand, the scaling function has been accessed numerically [51–53] and by means of an analytical mode coupling approach [54,55], based on a self-consistent one-loop calculation, i.e., to first order in λ.

In a recent letter, denoted in the following by II [56], we approached the strong coupling fixed point behavior from the point of view of the mapping of the Burgers equation onto an equivalent solid-on-solid or driven lattice gas model [11,57], which furthermore maps onto a discrete spin 1/2 chain model [58,59]. The quantum spin chain approach has been pro-

posed in [11,60], and considered further in [62], on the basis of the equivalence between the Liouville operator in the Master equation describing the evolution of the one-dimensional driven lattice gas, or the equivalent lattice interface solid-on-solid growth model, and a non-Hermitian spin 1/2 Hamiltonian. The quantum chain model has been treated by means of Bethe ansatz methods [11,60–63] and the dynamic exponent $z = 3/2$ obtained from the finite size mass gap scaling. In II we pushed the analysis further and, constructing a harmonic oscillator representation valid for large spin in combination with a continuum limit, we derived a Hamiltonian description and a set of coupled field equations of motion for the spin field, corresponding to the slope u, and a conjugate ''azimuthal'' angle field replacing the noise. The field equations admit spin wave solutions, corresponding to the linear diffusive modes and, more importantly, nonlinear localized soliton solutions, describing the growing steps in the original KPZ equation or the solitons or shocks in the Burgers equation. We also derived the soliton dispersion law and after a quasiclassical quantization identified in a heuristic manner the elementary excitations of the theory. From the dispersion law we deduced the dynamic exponent $z = 3/2$, characteristic of the zero temperature fixed point of the ''quantum theory.'' The picture that emerged from our analysis was that of a dilute quasiparticle gas of nonlinear soliton modes yielding $z = 3/2$ and a superposed spin wave gas, corresponding to $z = 2$, the dynamic exponent for the linear case. In II we also briefly discussed the operator algebra associated with the Hamiltonian representation and derived the field equations by means of a canonical representation of the Fokker-Planck equation for the equivalent Burgers equation. Whereas the Bethe ansatz investigations by their nature are restricted to special values of the coupling strength, corresponding to the fully asymmetric exclusion model [57], our analysis is valid for general coupling strength and thus constitutes an extension of the Bethe ansatz method to the general case of a continuum field theory. The analysis in II was in many respects incomplete and preliminary but it did indicate that the strong coupling fixed point behavior is intrinsically associated with the soliton modes in the Burgers equation since they both provide aspects of the growth morphology and also, independently, yield the dynamic exponent.

Here we present a *unified* approach to the noisy Burgers equation based on the Martin-Siggia-Rose (MSR) technique in functional form [65–70]. This method supersedes the analysis in II and does not make use of the mapping to a spin chain via a solid-on-solid model and therefore the implicit assumption of persistent universality classes under the mappings. Clearly, the different formulations are basically equivalent arising as they do from the same basic stochastic growth problem. The equivalence also indirectly demonstrates that the scaling properties of the different models, i.e., the solid-on-solid model, the spin chain, and the field formulation, fall in the same KPZ-Burgers universality class. The present functional path integral method, which also can be seen as a generalization of the stationary distribution (1.11) to the time-dependent nonlinear case, provides a many-body description of the morphology of a growing interface in terms of soliton excitations and also gives insight into the scaling behavior. Below we highlight some of our results.

(i) The path integral formulation yields a compact description of the noisy Burgers equation and provides expressions for the probability distributions and correlation functions. Reformulated as a canonical Feynman-type phase space path integral the approach allows for *a principle of least action*. Hence the weight of the different paths or interface configurations, corresponding to the noise-induced interface fluctuations, contributing to the path integral are controlled by an effective action. The role of the effective Planck constant is here played by the noise correlation strength Δ. The action in the path integral thus plays the same role for the dynamical configurations as the Hamiltonian in the Boltzmann factor for the static configurations in equilibrium statistical mechanics.

(ii) In the asymptotic weak noise limit the principle of least action implies that the dominant configurations arising from the solutions of the saddle point field equations correspond to a dilute nonlinear soliton gas with superposed linear diffusive modes. The canonical formulation and the principle of least action furthermore allow a dynamical description and associate energy, momentum, and action with the solitons and the diffusive modes.

(iii) The path integral formulation permits a ''quantum mechanical'' interpretation in terms of an underlying non-Hermitian relaxational ''quantum mechanics'' or ''quantum field theory.'' The noise-induced fluctuations here correspond to ''quantum fluctuations'' and the fluctuating growth morphology is described by a Landau-type quasiparticle gas of nonlinear ''quantum solitons'' and linear ''quantum diffusive modes.'' In the height field this corresponds to a morphology of growing steps with superposed linear modes. The ''quantum soliton'' dispersion law is gapless and characterized by an exponent 3/2; the ''quantum diffusive mode'' dispersion law is quadratic with a gap in the spectrum proportional to the soliton amplitude.

(iv) In the present formulation the scaling properties associated with the ''zero temperature'' fixed point in the underlying ''quantum field theory'' follow as a by-product from the soliton and diffusive mode dispersion laws and the spectral representation of the correlations. The dominant excitation in the long wavelength–low frequency limit identifies the relevant universality class. The present many-body formulation yields the known exponents. The dynamic exponents $z=3/2$ and $z=2$ are associated with the soliton and diffusive mode dispersion laws, respectively, whereas the roughness exponent $\zeta=1/2$ follows from a regularity property of the form factor in the spectral representation. The many-body formulation also explains the robustness of the roughness exponent under a change of universality class and provides a heuristic expression for the scaling function that has the same structure as the probability distribution for Lévy flights.

(v) From a field theoretical point of view we identify the noise strength Δ as the effective *small parameter*. Furthermore, the fundamental probability distribution or path integral has an *essential singularity* for $\Delta=0$. Hence our approach is based on a *nonperturbative saddle point or steepest descent approximation* to the path integral. Although the dynamic renormalization group method based on an expansion in the effective coupling strength $g=\lambda^2\Delta/\nu^3$ and thus in Δ yields the scaling exponents correctly and in combination

with a somewhat *ad hoc* mode coupling approach the scaling function, we believe that the saddle point contributions are essential ingredients in elucidating both the physical morphology and the scaling properties of the strong coupling fixed point.

The path integral representation of the noisy Burgers equation presented here is equivalent to a full-fledged one-dimensional non-Hermitian non-Lagrangian field theory and requires for its detailed analysis some advanced field theoretical techniques and methods from quantum chaos. In the present context we choose, however, a somewhat heuristic approach to the path integral in order to elucidate the emerging simple physical picture of a growing interface. This approach then also serves as a tutorial introduction to the field theoretical treatment to be presented elsewhere.

We also wish to comment on some recent work on the driven Burgers equation with noise at large length scales modeling forced turbulence. This problem has been treated using a variety of methods such as operator product expansions [71], instanton calculations [72–75], and replica methods [76]. In this context the nonperturbative instanton methods used in order to determine the tail of the velocity probability distribution are related to the present soliton approach in that they are also based on a saddle point approximation to the MSR functional integral and some of our results to be presented later have also been derived within the instanton approach.

The present paper is organized in the following way. In Sec. II we discuss the simple case of the linear Edwards-Wilkinson equation, mainly in order to emphasize the non-perturbative nature of the noise as regards the stationary driven regime. Since the soliton modes in the noisy Burgers equation turn out to be of crucial importance in understanding the morphology and scaling properties, we summarize in Sec. III the results obtained in I concerning the solitons and diffusive modes in the noiseless Burgers equation. In Sec. IV we set up the path integral formulation for the noisy Burgers equation in terms of the Martin-Siggia-Rose techniques in functional form. In Sec. V we perform a shift transformation of the path integral to a canonical Feynman path integral form and discuss the canonical structure and the associated symmetry algebra. Section VI is devoted to an asymptotic weak noise saddle point approximation and to the derivation of the deterministic coupled field equations replacing the Burgers equation. In Sec. VII we solve the field equations and derive nonlinear soliton and linear diffusive mode solutions. In Sec. VIII we discuss the dynamics of the solitons following from the principle of least action. The dominating morphology of a stochastically growing interface can be interpreted in terms of a dilute soliton gas; this aspect is discussed in some detail in Sec. IX. The fluctuation spectrum about the soliton solutions is basically given by the path integral, however, in Sec. X we take a heuristic point of view and discuss the fluctuations as ''quantum fluctuations'' in the underlying non-Hermitian ''quantum field theory.'' Section XI is devoted to a discussion of the scaling properties and universality classes on the basis of the ''elementary excitations'' in the ''quantum description.'' We also present a heuristic expression for the scaling function. Finally in Sec. XII we present a discussion and a conclusion.

II. THE LINEAR EDWARDS-WILKINSON EQUATION:
THE ROLE OF NOISE

Here we review the properties of the linear case described by the noise-driven Edwards-Wilkinson equation (1.6), in particular in order to elucidate the role of the noise. For the slope field this equation is given by

$$\frac{\partial u}{\partial t} = \nu \nabla^2 u + \nabla \eta \qquad (2.1)$$

with the noise η correlated according to Eq. (1.2), i.e.,

$$\langle \eta(x,t)\eta(x',t') \rangle = \Delta \delta(x-x')\delta(t-t'). \qquad (2.2)$$

Equation (2.1) has the form of a conservation law

$$\frac{\partial u}{\partial t} = -\nabla j \qquad (2.3)$$

with current

$$j = -\nu \nabla u - \eta. \qquad (2.4)$$

We note that with average vanishing ∇u at the boundaries the conservation law implies that the average off-set in the height, $\int \nabla h\, dx$, is conserved.

In wave number space, $u(k,t) = \int dx \, \exp(-ikx)u(x,t)$, and solving Eq. (2.1) as an initial value problem averaging over the noise according to Eq. (2.2) we obtain for the slope correlations

$$\langle u(k,t)u(-k,t') \rangle = [\langle u(k,0)u(-k,0) \rangle_i - (\Delta/2\nu)]$$
$$\times \exp[-(t+t')\nu k^2]$$
$$+ (\Delta/2\nu)\exp[-|t-t'|\nu k^2]. \quad (2.5)$$

Here $\langle \cdots \rangle_i$ denotes an average over initial values, which is assumed independent of the noise average $\langle \cdots \rangle$. The basic time scale is set by the wave number dependent lifetime $\tau(k) = 1/\nu k^2$, which diverges in the long wavelength limit $k \to 0$, characteristic of a conserved hydrodynamical mode. We note that at short times compared to $\tau(k)$, which sets the time scale for the transient regime, $\langle u(k,t)u(-k,t') \rangle$ is non-stationary and depends on the initial correlations, whereas at long times $t,t' \gg \tau(k)$ the correlations enter a stationary, time reversal invariant regime and depends only on $|t-t'|$. For vanishing initial slope, $u(k,0) = 0$, we obtain in particular the mean square slope fluctuations

$$\langle |u(k,t)|^2 \rangle = \frac{\Delta}{2\nu}\{1 - \exp[-2t/\tau(k)]\}, \qquad (2.6)$$

which approaches the saturation value $\Delta/2\nu$ for $t \gg \tau(k)$.

More precisely, it follows from Eq. (2.5) that for fixed $t - t'$ the transient term can be neglected at times greater than a characteristic crossover time t_{co} of order

$$t_{co} \sim (1/\nu k^2)\ln(1/\Delta), \qquad (2.7)$$

depending also on the noise strength Δ. This time thus defines the onset of the stationary regime. For $t \gg \tau(k)$, t_{co} noise-induced fluctuations built up and the mean square

slope fluctuations approach the constant value $\Delta/2\nu$. For $\Delta \to 0$, $t_{co} \to \infty$ and the system never enters the stationary regime.

The ''elementary excitation'' is the diffusive mode $u(k,t) \propto \exp(\pm \nu k^2 t)$. In frequency space $u(k,\omega) = \int dt \, \exp(i\omega t)u(k,t)$ and the slope correlation function assumes the Lorentzian diffusive form, characteristic of a hydrodynamical mode,

$$\langle u(k,\omega)u(-k,-\omega) \rangle = \frac{\Delta k^2}{\omega^2 + (\nu k^2)^2}, \qquad (2.8)$$

with diffusive poles at $\omega_k^0 = \pm i\nu k^2$, a strength given by $\Delta/\nu^2 k^2$ and a linewidth νk^2. We note that in the stationary regime both the decaying and growing modes, $u \propto \exp(\pm \nu k^2 t)$, contribute to the stationary correlations. Time reversal invariance is thus induced from the microscopic reversibility of the noise-driven system. In the transient regime for $t \ll \tau(k)$ the initial conditions enter and we must choose the solution propagating forward in time, $u \propto \exp(-\nu k^2 t)$, in order to satisfy causality.

From Eq. (2.8) we also obtain the scaling function

$$f(w) = (\Delta/2\nu)(4\pi\nu)^{-1/2}w^{-1/2}\exp[-1/4\nu w], \quad (2.9)$$

in accordance with the general form in Eq. (1.7) yielding the EW exponents in Table I defining the EW universality class. For large w $f(w) \sim w^{-1/2}$; for small w $f(w) \to 0$ but with an essential singularity for $w = 0$. In frequency–wave number space the scaling form is

$$\langle u(k,\omega)u(-k,-\omega) \rangle = k^{-2}g(\omega/k^2) \qquad (2.10)$$

and we directly infer the scaling function

$$g(w) = \frac{\Delta}{\nu + w^2}. \qquad (2.11)$$

In Fig. 2 we have shown the slope correlation function and the scaling functions f and g in the EW case.

In contrast to the noisy Burgers equation, the EW equation does not provide a proper description of a growing interface. This is seen by expressing Eq. (2.1) in the form

$$\frac{\partial u}{\partial t} = \nu \nabla^2 \frac{\delta F}{\delta u} + \nabla \eta, \qquad (2.12)$$

where the effective free energy is given by

$$F = \frac{1}{2}\int dx\, u^2. \qquad (2.13)$$

Using the fluctuation-dissipation theorem to relate the noise strength to an effective temperature T it then follows that the EW equation describes time-dependent fluctuations in an *equilibrium* system with temperature $T = \Delta/2\nu$ and with an equilibrium distribution given by the Boltzmann factor in Eq. (1.11), i.e.,

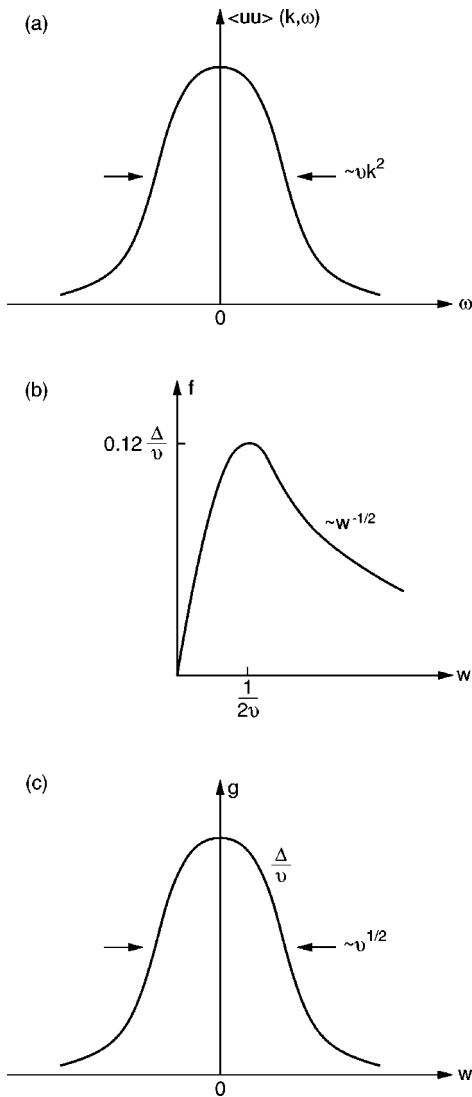$$P(u) \propto \exp\left[-\frac{2\nu}{\Delta}F\right]. \qquad (2.14)$$

FIG. 2. In (a) we depict the slope correlation function for the diffusive mode in the linear EW case. The Lorentzian is centered about $\omega=0$ and has the ''hydrodynamical'' line width νk^2 vanishing in the long wavelength limit. In (b) we show the scaling functions f for the space and time-dependent slope correlations. For large w f falls off as $w^{-1/2}$, for small w $f\to 0$ with an essential singularity for $w=0$. f is peaked at the value $\sim 0.12\Delta/\nu$ for $w=1/2\nu$. In (c) we have shown the scaling function g for the wave number–frequency-dependent correlations. g has a Lorentzian form with height Δ/ν and width $\sim \nu^{1/2}$ (arbitrary units).

We note here in the linear case that the noise strength Δ seems to play a special role. Whereas Δ enters linearly in the correlation function $\langle uu\rangle(k,\omega)$, the limit of vanishing noise strength, $\Delta\to 0$, appears as an *essential singularity* in the stationary distribution (2.14). Since the distribution $P(u)$ appropriately generalized to the time-dependent case is the generator for the correlation function $\langle uu\rangle$ and higher cumulants, it is clearly the fundamental object and the role of the noise strength Δ as a nonperturbative parameter an important observation. More precisely the point is the following: Whereas the damping constant ν together with the relevant wave number k define the time scale for the transient regime where the system has memory and evolves forward in time in an irreversible manner, the presence of the noise is essen-

tial in order for the system to leave the transient regime *at all* and to enter the stationary regime where the system is time reversal invariant, as, for example, reflected in the evenness in ω in the slope correlation function (2.8). In the absence of the noise the system simply decays owing to dissipation unless it is driven by deterministic currents at the boundaries. Imposing the noise and driving the system stochastically is thus a *singular* process, as reflected mathematically by the essential singularity in the distribution (2.14).

Although the above observation of the nonperturbative role of the noise strength Δ is a trivial statement in the linear case where is just reflects the structure of the Boltzmann factor, we will later show that in a more complete theory of a growing interface, described by the nonlinear noisy Burgers equation, it is essential to take into account nonperturbative contributions in the noise strength Δ.

III. THE SOLITON MODE IN THE NOISELESS BURGERS EQUATION

It turns out that the soliton excitation in the *noiseless* Burgers equation when properly generalized play an important role in the understanding of the growth morphology and strong coupling behavior of the noisy Burgers equation. In I we discussed in some detail the soliton and diffusive mode solutions in the noiseless Burgers equation and performed a linear stability analysis. Here we briefly summarize those aspects of the analysis in I that will be of importance in the discussion of the noisy case.

The noiseless or deterministic Burgers equation has the form [3–6]

$$\frac{\partial u}{\partial t}=\nu\nabla^2 u+\lambda u\nabla u, \tag{3.1}$$

and is a nonlinear diffusive evolution equation with a linear term controlled by the damping or viscosity ν and a nonlinear mode coupling term characterized by λ. In the context of fluid motion the nonlinear term gives rise to convection as in the Navier Stokes equation; for an interface the term corresponds to a slope-dependent growth.

Under time reversal $t\to -t$ and the transformation $u\to -u$ the equation is invariant provided $\nu\to -\nu$. This indicates that the linear diffusive term and the nonlinear convective or growth term play completely different roles. The diffusive term is intrinsically irreversible whereas the growth term corresponding to a mode coupling leads to a cascade in wave number space and generates genuine transient growth. The transformation $t\to -t$ is absorbed in $u\to -u$ or, alternatively, $\lambda\to -\lambda$, corresponding to a change of growth direction. We also note that the equation is invariant under the parity transformation $x\to -x$ provided $u\to -u$. This feature is related to the presence of a single spatial derivative in the growth term and implies that the equation only supports solitons or shocks with one parity, that is *right hand* solitons. Finally, the Burgers equation is invariant under the Galilean symmetry group (1.8) and (1.9), that is a Galilean boost to a frame moving with velocity λu_0 is absorbed by a shift in the slope field by u_0.

The irreversible and diffusive structure of Eq. (3.1) implies that an initial disturbance eventually decays owing to

the damping term $\nu\nabla^2 u$. In the linear case the slope field decays by simple diffusion $u(x,t)\propto\exp(-\nu k^2 t)\exp(\pm ikx)$ as discussed in Sec. II. In the presence of the nonlinear mode coupling term the equation also supports localized soliton or kink profiles [77–80] with given parity. In the static case the symmetric positive parity or *right hand* soliton has the form

$$u(x)=u_+ \ \tanh[k_s(x-x_0)], \qquad (3.2)$$

$$k_s=\lambda u_+/2\nu. \qquad (3.3)$$

We have introduced the characteristic wave number k_s setting the inverse length scale associated with the static soliton, x_0 denotes the center of mass position. The width of the soliton is of order $1/k_s$ and depends on the amplitude u_+. In the inviscid limit $\nu\to 0$ or for strong coupling $\lambda\to\infty$, the wave number $k_s\to\infty$ and the soliton reduces to a sharp shock wave discontinuity.

Boosting the static soliton in Eq. (3.2) to a finite propagation velocity v and at the same time shifting u we obtain, denoting the right and left boundary values by $u_\pm$, the soliton solution

$$u(x,t)=\frac{u_+ + u_-}{2}+\frac{u_+ - u_-}{2}$$
$$\times\tanh\left[\frac{\lambda}{4\nu}(u_+ - u_-)(x-vt-x_0)\right] \qquad (3.4)$$

with velocity v given by the soliton condition

$$u_+ + u_- = -\frac{2v}{\lambda}. \qquad (3.5)$$

We note that the soliton condition (3.5) is consistent with the fundamental Galilean invariance and remains invariant under the transformation $v\to v+\lambda u_0$ and $u_\pm\to u_\pm - u_0$. Also, unlike the case for the Lorentz invariant ϕ^4 and sine-Gordon evolution equations [77], the propagation velocity in the present case is tied to the amplitude boundary values of the soliton—a feature of the Galilean invariance. In Fig. 3 we have depicted the *right hand* soliton solution given by Eq. (3.4) and the associated height field h.

In the linear case for $\lambda=0$ case the diffusive modes with dispersion

$$\omega_k^0=-i\nu k^2 \qquad (3.6)$$

"exhaust" the spectrum of relaxational modes. For $\lambda\neq 0$ the soliton profile acts as a reflectionless Bargman potential giving rise to a bound state at zero frequency, corresponding to the translation mode of the soliton—the Goldstone mode restoring the broken translational invariance, and a band of phase-shifted diffusive scattering modes [80]. The resulting change of density of states is in accordance with Levinson's theorem in that the potential traps a bound state and depletes the continuum of one state. In the presence of the soliton the diffusive modes furthermore develop a gap in the spectrum of ω_k as depicted in Fig. 4,

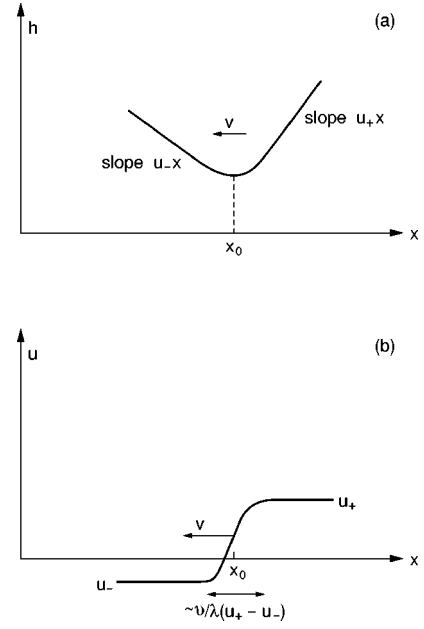$$\omega_k=-i\nu(k^2+k_s^2). \qquad (3.7)$$

FIG. 3. We show a single moving soliton profile propagating to the left and the corresponding smoothed cusp in the growth profile. This configuration is driven by currents at the boundaries, corresponding to nonvanishing $u_\pm$ and is persistent in time (arbitrary units).

An asymptotic analysis of the noiseless Burgers equation in the inviscid limit $\nu\to 0$ [81] shows that an initial configuration breaks up into a "gas" of propagating and coalescing kinks connected by ramp solutions of the form $u\propto$const$-x/\lambda t$. This allows for the following qualitative picture of the transient time evolution: Although the nonlinear mode coupling term is incompatible with a proper superposition principle we can still along the lines of the evolution of integrable one dimensional evolution equations [77] envisage that an initial configuration "contains" a number of *right hand* solitons connected by ramps. In the course of time the solitons propagate and coalesce. Superposed on the soliton gas is a gas of phase-shifted diffusive modes. As discussed in I the gap in the diffusive spectrum can be associated with the current flowing towards the center of the solitons. The damping of the configuration predominantly takes place at the center of the soliton where u varies rapidly
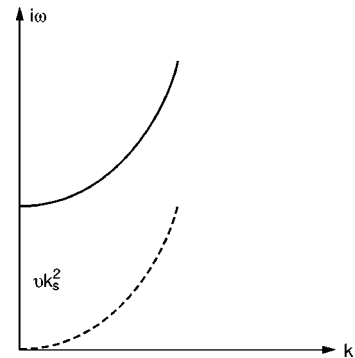

FIG. 4. We show the diffusive dispersion law in the presence of a soliton. The gap in the spectrum is given by $\nu k_s^2=\lambda^2 u_+^2/4\nu$ where u_+ is the soliton amplitude. The dashed line indicates the gapless spectrum in the linear case (arbitrary units).
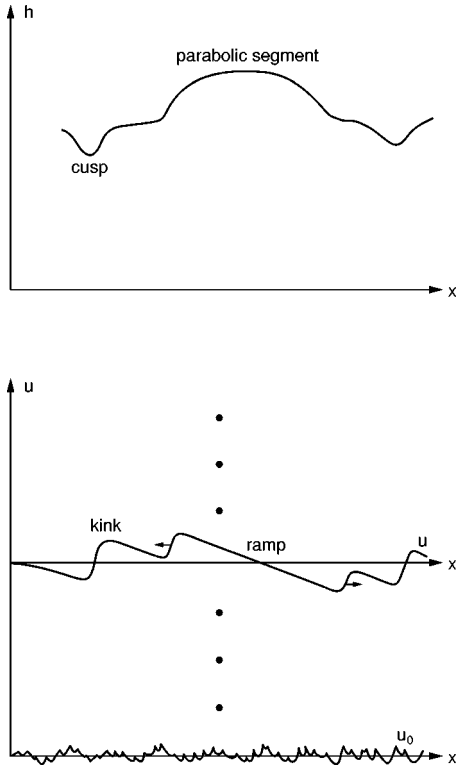
FIG. 5. We here depict the transient evolution of the slope field u from an initial configuration u_0 in the case of the noiseless Burgers equation. We have also shown the evolution of the associated height field h. The transient morphology consists of propagating *right hand* solitons connected by ramps (arbitrary units).

thus enhancing the damping term $\nu\nabla^2 u$. We also note that only parity breaking *right hand* solitons are generated in the noiseless Burgers equation. In Fig. 5 we have shown the transient evolution of the slope field and the associated height field.

IV. PATH INTEGRAL REPRESENTATION OF THE NOISY BURGERS EQUATION

In this section we begin the analysis of the noisy Burgers equation. In our discussion in Sec. II of the linear EW equation we noticed that the noise strength Δ enters in a nonperturbative way in the stationary distribution in Eq. (1.11). Whereas this, of course, is a trivial observation in the linear case since the EW equation describes fluctuations in equilibrium and $\Delta \propto T$, that is the singularity structure is the same as the low-T limit of the Boltzmann factor $\exp(-E/T)$, the presence of the nonlinear mode coupling growth term in the noisy Burgers equation renders the situation much more subtle. We are now dealing with an intrinsically nonequilibrium situation. The noise drives the system into a far-from-equilibrium stationary state and equilibrium statistical mechanics does not apply. On the other hand, from our study of the noiseless Burgers equation, we have learned that the soliton excitations play an important role in the dynamics of the morphology of a growing interface and is a direct signature of the nonlinearity. The issue facing us is then how to include both the nonperturbative aspects of the noise and the nonlinear soliton structure in a consistent way. It turns out that the functional formulation of the Martin-Siggia-Rose

techniques provides the appropriate formal and practical language for such an approach [65–70].

Our starting point is the noisy Burgers equation (1.1) for the fluctuating slope field u, i.e.,

$$\frac{\partial u}{\partial t} = \nu\nabla^2 u + \lambda u\nabla u + \nabla\eta, \qquad (4.1)$$

which has the structure of a conserved nonlinear Langevin equation with current

$$j = -\nabla u - \frac{\lambda}{2}u^2 - \eta. \qquad (4.2)$$

For the noise we assume a Gaussian distribution

$$P(\eta) \propto \exp\left[-\frac{1}{2\Delta}\int dx\, dt\,\eta(xt)^2\right], \qquad (4.3)$$

where η is correlated according to Eq. (1.2), i.e.,

$$\langle\eta(x,t)\eta(x',t')\rangle = \Delta\delta(x-x')\delta(t-t'). \qquad (4.4)$$

Unlike the transient relaxation of an initial value configuration described by the deterministic Burgers equation, the noisy Burgers equation is driven continuously by the conserved noise $\nabla\eta$, corresponding to a fluctuating component of the current j in Eq. (4.2). Energy is fed into the system via the noise and dissipated by the linear damping term. The nonlinear mode coupling gives rise to a cascade in wave number space corresponding to ''dissipative structures'' in the growth morphology. This mechanism changes the probability distributions and associated correlations (moments), scaling exponents, and scaling functions from the EW case in Sec. II. In other words, Eq. (4.1) acts as a nonlinear box that transforms the input noise $\nabla\eta$ to an output slope field u.

In the Martin-Siggia-Rose techniques the probability distribution for the slope field $P(u)$ and the correlations $\langle uu\rangle_\eta$ are conveniently derived from an effective partition function or generator [42]

$$Z(\mu) = \left\langle\exp\left[i\int dx\, dt\, u(x,t)\mu(x,t)\right]\right\rangle_\eta. \qquad (4.5)$$

Here $\mu(x,t)$ is a generalized chemical potential or external conjugate field coupling to the slope field $u(x,t)$ and $\langle\cdots\rangle_\eta$ denotes an average over the input noise η, implementing the nonlinear stochastic relationship provided by the Burgers equation (4.1). In terms of Z we have, for example, the probability distribution $P(u) = \langle\delta(u-u(x,t))\rangle_\eta$,

$$P(u(x,t)) = \int\prod_{xt} d\mu$$
$$\times\exp\left[-i\int dx\, dt\, u(x,t)\mu(x,t)\right]$$
$$\times Z(\mu(x,t)) \qquad (4.6)$$

and the correlation function

$$\langle u(x,t)u(x',t')\rangle = -\left[\frac{\delta Z(\mu)}{\delta\mu(x,t)\,\delta\mu(x',t')}\right]_{\mu=0}; \quad (4.7)$$

higher moments are derived in a likewise manner. In order to incorporate the nonlinear constraint imposed by the Burgers equation we insert the identity

$$\int \prod_{xt} du\,\delta\left(\frac{\partial u}{\partial t} - \nu\nabla^2 u - \lambda u\nabla u - \nabla\eta\right) = 1 \quad (4.8)$$

in the partition function $Z(\mu)$; for a first order evolution equation one can show that causality implies that the Jacobian relating du to $\partial u/\partial t$ equals unity [82]. Finally, exponentiating the delta function constraint in Eq. (4.8) and averaging over the noise distribution according to Eq. (4.3) we obtain

$$Z(\mu) = \int \prod_{xt} du\,dp\, \exp[iG]\exp\left[i\int dxdtu\mu\right], \quad (4.9)$$

where the effective functional G is given by

$$G = \int dxdt\left[p\left(\frac{\partial u}{\partial t} - \nu\nabla^2 u - \lambda u\nabla u\right) + \frac{i}{2}\Delta(\nabla p)^2\right]. \quad (4.10)$$

The path or functional integral (4.9) with G given by Eq. (4.10) effectively replaces the stochastic Burgers equation (4.1). The path integral is deterministic and the noise η is replaced by the different configurations or paths contributing to Z. In this sense Z is an effective partition function for the dynamical problem and G an effective Hamiltonian, analogous to the Hamiltonian in the partition function $Z = \Sigma \exp(-H/T)$ in equilibrium statistical mechanics. We also note that the transcription of the Burgers equation to a path integral leads to the appearance of an additional noise field p, arising from the exponentiation of the delta function constraint in Eq. (4.8) [65–70], and replacing the stochastic noise in Eq. (4.1).

Since the path integral formulation provides a field theoretical framework allowing for functional and diagrammatic techniques, Feynman rules, skeleton graphs, Ward identities, etc., it is mostly used in order to generate perturbation expansions in powers of the nonlinear coupling $\lambda u\nabla u$, forming the basis for the dynamic renormalization group method [42–44,55,83–85]. It is, however, worthwhile noting that such a field theoretic expansion has precisely the same structure as the one produced by directly iterating the Burgers equation (4.1) in powers of $\lambda u\nabla u$ and averaging over the noise term by term according to Eq. (4.4).

Also, corroborating our remarks in Sec. II, we notice from the structure of the path integral (4.9) and (4.10) that the noise strength Δ appears as a singular parameter in the sense that $\Delta\rightarrow 0$ gives rise to the singular delta function constraint for the Burgers equation. This limit is, however, much more transparent when we express Z in a ''canonical form.''

V. CANONICAL TRANSFORMATION TO A HAMILTONIAN FORM: PHASE SPACE PATH INTEGRAL SYMMETRIES

By inspection of the path integral in Eqs. (4.9) and (4.10) we notice that it has the same structure as the usual phase space Feynman path integral [42] as regards the kinetic term $p\,\partial u/\partial t$ in F but that otherwise p and u do not appear in a canonical combination. This situation can, however, be remedied by performing a simple complex shift of the noise variable p,

$$p = \frac{\nu}{\Delta}(iu - \varphi), \quad (5.1)$$

in Eqs. (4.9) and (4.10). The partition function $Z(\mu)$ can then be written as

$$Z(\mu) = \text{const}\times \int \prod_{xt} du\,d\varphi\, \exp\left[i\frac{\nu}{\Delta}S\right]\exp\left[i\frac{\nu}{\Delta}S_B\right]$$
$$\times \exp\left[i\int dxdtu\mu\right], \quad (5.2)$$

where the surface contribution S_B has the form

$$S_B = \int dxdt\left\{\frac{\partial}{\partial t}\left[\frac{i}{2}u^2 - u\varphi\right] + \nabla\right.$$
$$\left.\times\left[\nu(\varphi - iu)\nabla u + \frac{\lambda}{2}\varphi u^2 - i\frac{\lambda}{3}u^3\right]\right\}. \quad (5.3)$$

However, assuming that the path integral operates in a space time LT box, i.e., $|x|<L/2$ and $|t|<T/2$, and imposing, for example, vanishing boundary conditions for u in both space and time the surface contribution S_B to the action vanishes identically.

The bulk contribution to the action is given by the canonical form [86]

$$S = \int dxdt\left[u\frac{\partial\varphi}{\partial t} - \mathcal{H}(u,\varphi)\right] \quad (5.4)$$

with the complex Hamiltonian density

$$\mathcal{H} = -i\frac{\nu}{2}[(\nabla u)^2 + (\nabla\varphi)^2] + \frac{\lambda}{2}u^2\nabla\varphi. \quad (5.5)$$

The Hamiltonian density consists of two terms: A relaxational or irreversible harmonic component, $-i(\nu/2)[(\nabla u)^2 + (\nabla\varphi)^2]$, corresponding to the diffusive aspects of a growing interface, i.e., the linear damping, and a nonlinear reversible mode coupling component, $(\lambda/2)u^2\nabla\varphi$, associated with the drive λ.

One feature of the transcription of the noisy Burgers equation to a canonical path integral form is that the effective Hamiltonian density (5.5) driving the dynamics of the system is in general complex. This particular aspect was also encountered in the treatment in II where the growth term in the spin chain Hamiltonian turned out to be complex; see also Ref. [57]. We also notice that the doubling of variables, i.e., the replacement of the stochastic noise η by an addi-

tional noise field φ in the path integral, was also encountered in the treatment in II in the canonical oscillator representation of the spin variables.

It is here instructive to compare the above path integral for the relaxational growth dynamics of the Burgers equation with the usual phase space path integral formulation in quantum mechanics or quantum field theory [42]. Here the partition function has the form

$$Z = \int \prod_{xt} dp \, dq \, \exp\left[\frac{i}{\hbar} S\right] \tag{5.6}$$

with the classical action

$$S = \int dx \, dt \left[p \frac{\partial q}{\partial t} - \mathcal{H}(p,q) \right], \tag{5.7}$$

where p and q are considered canonically conjugate variables and $\mathcal{H}(p,q)$ the usual classical Hamiltonian density.

Comparing Eqs. (5.6) and (5.7) with Eqs. (5.2)–(5.4) it is evident that the structures of the two path integral formulations are quite similar and we are led to identify the noise strength Δ/ν with an ''effective'' Planck constant and the Hamiltonian density $\mathcal{H}$ as the generator of the dynamics. The classical limit thus corresponds to the weak noise limit $\Delta \to 0$ and in analogy with the quasiclassical or WKB approximation in quantum mechanics, $\Delta \to 0$, constitutes a singular limit in accordance with our previous remarks. The partition function Z with the action S given by Eqs. (5.2)–(5.5) thus constitutes the required generalization of the stationary distribution $P(u) \propto \exp[-(\nu/\Delta)\int dx u^2]$ in Eq. (1.11) to the time-dependent case [87]. By comparison we furthermore conclude that the slope field u and the noise field φ, replacing the Gaussian noise in Eq. (4.1), are canonically conjugate momentum and coordinate variables satisfying the Poisson bracket algebra [88]

$$\{u(x), \varphi(x')\} = \delta(x - x') \tag{5.8}$$

and that the Hamiltonian or energy

$$H = \int dx \, \mathcal{H} = \int dx \left[-i \frac{\nu}{2}[(\nabla u)^2 + (\nabla \varphi)^2] + \frac{\lambda}{2} u^2 \nabla \varphi \right] \tag{5.9}$$

is the generator of time translations according to the equations of motion

$$\frac{\partial u}{\partial t} = \{H, u\}, \tag{5.10}$$

$$\frac{\partial \varphi}{\partial t} = \{H, \varphi\}. \tag{5.11}$$

Drawing on the mechanical analog the momentum P, the generator of translations in space, is also easily identified from the basic transformation properties,

$$\nabla u = \{P, u\}, \tag{5.12}$$

$$\nabla \varphi = \{P, \varphi\}, \tag{5.13}$$

and it follows that

$$P = \int dx \, g, \tag{5.14}$$

$$g = u \nabla \varphi, \tag{5.15}$$

where g is the momentum density.

In order to elucidate the canonical structure of the path integral (5.2)–(5.5) and the analogy with the usual phase space Feynman path integral we have generated a complex Hamiltonian (5.5). Note, however, that by formally rotating the noise field in phase space $\varphi \to i\varphi$ the Hamiltonian and the action become purely imaginary, leading to a real path integral.

The symmetries discussed in the context of the quantum spin chain representation in II are also easily recovered here. Noting that $\mathcal{H}$ is invariant under a constant shift of the noise field, $\varphi \to \varphi + \varphi_0$, we infer that the integrated slope field

$$M = \int dx \, u, \tag{5.16}$$

i.e., the total offset of the height field, $h = \int dx \, u$, across the interface, is a constant of motion,

$$\{H, M\} = 0. \tag{5.17}$$

This is consistent with the local conservation law following from the structure of the Burgers equation, but is here a consequence of the structure of the path integral. The invariance under a shift of φ is equivalent to the invariance of the Burgers equation under a shift of the noise η in the noise term $\nabla \eta$. Similarly, under a constant shift of the slope field $u \to u + u_0$, we have, introducing the momentum density g, $\mathcal{H} \to \mathcal{H} + \lambda u_0 g + (\lambda/2)u_0^2 \nabla g$, or since the last term is a total derivative, $H \to H + \lambda u_0 P$, corresponding to an associated Galilean transformation with velocity $-\lambda u_0$. For the integrated noise field

$$\Phi = \int dx \, \varphi \tag{5.18}$$

we thus obtain the Poisson bracket algebra

$$\{H, \Phi\} = \lambda P, \tag{5.19}$$

which together with

$$\{H, M\} = \{H, P\} = 0, \tag{5.20}$$

$$\{P, \Phi\} = \{P, M\} = 0, \tag{5.21}$$

and

$$\{\Phi, M\} = L, \tag{5.22}$$

defines the symmetry algebra. We note again that the nonlinear coupling strength λ enters the Poisson bracket (5.19) and thus is a structural constant of the symmetry group.

We finally wish to comment on the properties of the path integral in Eqs. (5.2)–(5.5) under time reversal $t \to -t$. By construction the path integral applies at late times compared to any initial time t_0, defining the initial value of the slope

configuration u_0. This implies that the noise in the Burgers equation has driven the system into a stationary time regime and that the transients associated with u_0 have died out. In the linear case for $\lambda = 0$, we note by inspection that the path integral is invariant under the combined operation $t \to -t$ and $\varphi \to -\varphi$, implying that the slope correlations are not only stationary but also invariant under time reversal. This is in agreement with the analysis of the noisy EW equation in Sec. II where we obtained $\langle uu \rangle(k,\omega) = \Delta k^2 / [\omega^2 + (\nu k^2)^2]$, implying that $\langle uu \rangle(x,t)$ depends on $|t|$. This is consistent with the description of an equilibrium interface and is just an expression of microscopic reversibility. In the presence of the drive for $\lambda \neq 0$ the path integral is invariant under the combined transformation $t \to -t$, $\varphi \to -\varphi$, and $\lambda \to -\lambda$ showing that the term $(\lambda/2) u^2 \nabla \varphi$ in the Hamiltonian (5.9) gives rise to a proper growth direction thereby breaking time reversal invariance, that is, we are dealing with a genuine nonequilibrium phenomenon.

VI. FIELD EQUATIONS IN THE WEAK NOISE LIMIT: SADDLE POINT APPROXIMATION

The basic structure of the path integral (5.2)–(5.5) is illustrated by the simple one-dimensional integral

$$I(\Delta) = \int du \, \exp\left[i \frac{1}{\Delta} S(u) \right] \exp[i\mu u], \qquad (6.1)$$

where Δ is the small parameter (the noise strength). In the limit $\Delta \to 0$ the integral $I(\Delta)$ is approximated by a steepest descent calculation, which amounts to an expansion of $S(u)$ about an extremum u_0, $S(u) \sim S(u_0) + \frac{1}{2} S''(u_0)(u - u_0)^2$ and a subsequent calculation of a Gaussian integral. For small Δ we then obtain

$$I(\Delta) = \exp\left[i \frac{1}{\Delta} S(u_0) \right] \exp[i\mu u_0] \exp\left[-i\mu^2 \frac{\Delta}{2 S''(u_0)} \right]$$
$$\times \left[\frac{-2\pi i \Delta}{S''(u_0)} \right]^{1/2}. \qquad (6.2)$$

The leading contribution to $I(\Delta)$ is given by $\exp[iS(u_0)/\Delta]$ and is thus determined by the extremal value of the action. This part, however, goes along with a multiplicative factor, $[-2\pi i \Delta / S''(u_0)]^{1/2}$, arising from the Gaussian integral sampling the fluctuations about the stationary points; this term is the first in an asymptotic expansion in powers of $\Delta^{1/2}$. We notice that there is an essential singularity for $\Delta = 0$, signaling the nonperturbative aspects of a steepest descent calculation; the result cannot be obtained as a perturbation expansion in powers of Δ. The analysis of the path integral now essentially follows the same procedure but is rendered much more difficult owing to the field theoretical phase space structure of the problem. In Fig. 6 we have depicted the principle of a saddle point or steepest descent calculation of $I(\Delta)$.

In the weak noise limit $\Delta \to 0$ the asymptotically leading contribution to the path integral thus arises from configurations or paths (u, φ) corresponding to an extremum or stationary point of the action S. Invoking the variational condition $\delta S = 0$ with respect to independent variations of the
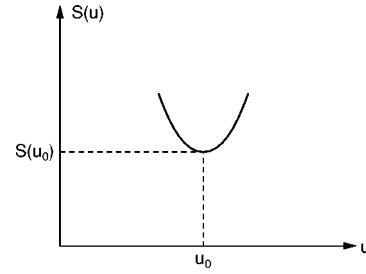

FIG. 6. Here we depict in graphic form the basic principle of an asymptotic steepest descent or saddle point calculation. The leading contribution to the integral $I(\Delta)$ in Eq. (6.1) arises from the saddle point u_0 (in the figure a minimum) and nearby fluctuations $\delta u = u_0 - u$ (arbitrary units).

slope field u and the canonically conjugate noise field φ, δu and $\delta \varphi$, with vanishing variations at the boundaries of the space time LT box, we readily infer, using Eq. (5.9), the classical equations of motion [86]

$$\frac{\partial u}{\partial t} = -\frac{\delta H}{\delta \varphi} = \{H, u\}, \qquad (6.3)$$

$$\frac{\partial \varphi}{\partial t} = +\frac{\delta H}{\delta u} = \{H, \varphi\}. \qquad (6.4)$$

Implementing the functional derivation or, equivalently, using the Poisson bracket relation (5.8), we obtain

$$\frac{\partial u}{\partial t} = -i\nu\nabla^2\varphi + \lambda u \nabla u, \qquad (6.5)$$

$$\frac{\partial \varphi}{\partial t} = +i\nu\nabla^2 u + \lambda u \nabla \varphi. \qquad (6.6)$$

The above coupled field equations (6.5) and (6.6) are a fundamental result of the present analysis. They provide a *deterministic* description of the noisy Burgers equation in the asymptotic nonperturbative weak noise limit. The equations have the same form as the ones derived in II based on the quasiclassical limit of the quantum spin chain representation. Furthermore, the parameter identification is in accordance with the ''quantum representation'' of the Fokker-Planck equation. As regards the considerations in II this demonstrates that the precise identification of the quasiclassical limit is in fact a weak noise limit in the exact path integral representation of the Burgers equation.

First of all we observe that the field equation for the slope field u has the form of a conservation law, $\partial u / \partial t = -\nabla u$, with current $j = -(\lambda/2)u^2 + i\nu\nabla\varphi$. The fluctuating component in the current in the noisy Burgers equation, $j = i(\lambda/2)u^2 - \nu\nabla u - \eta$, is thus replaced by the noise field φ and admissible solutions must yield an imaginary noise field in order to render a real current, corroborating our remarks in the previous section. The field equation for the noise field is parametrically coupled to the slope field and in the presence of the coupling λ driven by the momentum density $g = u\nabla\varphi$.

Secondly, we confirm that the field equations are invariant under the Galilean transformation (1.8) and (1.9),

$$u(x,t) \rightarrow u(x - \lambda u_0 t, t) - u_0, \qquad (6.7)$$

$$\varphi(x,t) \rightarrow \varphi(x - \lambda u_0 t, t) \qquad (6.8)$$

and under an arbitrary shift in φ

$$\varphi(x,t) \rightarrow \varphi(x,t) - \varphi_0, \qquad (6.9)$$

in accordance with the general discussion of the symmetry algebra and consistent with the symmetry properties of the noiseless and noisy Burgers equations.

As mentioned in Sec. I the instanton approach to driven turbulence in the Burgers equation is also based on a saddle point approximation to the Martin-Siggia-Rose functional in Eq. (4.10). In particular, applying the shift transformation (5.1) to the saddle point equations in Ref. [72], also discussed in Refs. [75], and considering short range correlated noise, one obtain the field equations (6.5) and (6.6). Also, the expressions for the energy and momentum in Eqs. (5.9) and (5.14) were discussed in Refs. [72–75].

One final comment on the classical zero noise limit. We maintain that in the asymptotic nonperturbative weak noise limit the coupled field equations provide the correct description of the leading behavior of the noisy Burgers equation. In order to obtain the noiseless Burgers equation discussed in Sec. III we must confine the noise field strictly to the line $\varphi = iu$ in (u, φ) phase space in which case both field equations reduce to the noiseless Burgers equation (3.1). Note, however, that setting $\varphi = -iu$ we obtain the noiseless Burgers equation with ν replaced by $-\nu$ supporting *growing* linear modes and a *left hand* nonlinear soliton mode—the missing modes necessary in order to describe the correct morphology in the noise-driven stationary regime. In other words, we anticipate that the lines $\varphi = \pm iu$ define regions for the stationary steepest descent or saddle point solutions of the field equations. The vicinity of these lines corresponds to the Gaussian fluctuations about the stationary points, that is the linear diffusive modes. This picture will in fact be borne out when we turn to an analysis of the field equations in the next section. In Fig. 7 we have shown the extremal paths, corresponding to the saddle point solutions and the nearby paths characterizing the fluctuations in (u, φ) phase space.

VII. SOLITON AND DIFFUSIVE MODE SOLUTIONS OF THE FIELD EQUATIONS

The field equations (6.5) and (6.6) for u and φ constitute a set of nonlinear coupled partial differential equations. The general solution is not known. Unlike the noiseless Burgers equation, which can be solved by means of the nonlinear Cole-Hopf transformation, similar substitutions do not seem to work for the field equations. Presently, it is not known whether the field equations belong to the small class of nonlinear evolution equations, which can be integrated partly or completely by means of the inverse scattering method and related techniques. We are therefore obliged to choose a more pedestrian approach and search for special solutions to the equations [77,78,80].
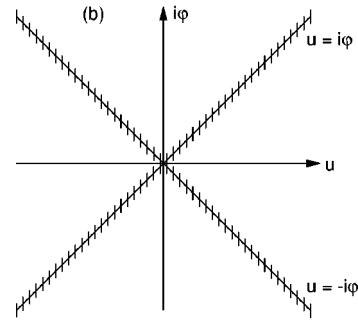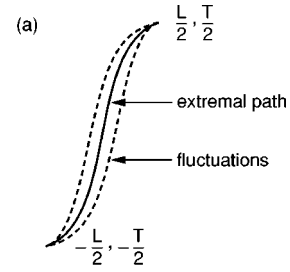

FIG. 7. In (a) we show the ''classical'' path corresponding to the stationary point of the action S in the weak noise limit and nearby paths corresponding to fluctuations. In (b) we show the saddle point regions in (u, φ) phase space (arbitrary units).

A. Stationary states

We note that the constant slope-constant noise configurations

$$u = u_0, \qquad (7.1)$$

$$\varphi = \varphi_0 \qquad (7.2)$$

are trivial saddle point solutions with vanishing energy, momentum, and action. They form an infinitely degenerate set and correspond to the zero-energy aligned ferromagnetic spin states discussed in paper II. The degenerate stationary slope configurations are related by a Galilean transformation and we shall in general choose a state with vanishing slope corresponding to a horizontal interface. As regards the noise field we are free to choose it equal to zero. In the phase space plot in Fig. 7 the background stationary state or the ''vacuum'' thus corresponds to the origin $(u, \varphi) = (0,0)$.

B. Linear diffusive modes

In the linear case for $\lambda = 0$ the Hamiltonian H is harmonic in the fields u and φ. The coupled field equations are linear,

$$\frac{\partial u}{\partial t} = -i\nu \nabla^2 \varphi, \qquad (7.3)$$

$$\frac{\partial \varphi}{\partial t} = +i\nu \nabla^2 u, \qquad (7.4)$$

and describe the weak noise limit of the EW equation. Expanding about the stationary state $(u_0, \varphi_0) = (0,0)$ the equations readily admit the solutions

$$u(xt) = \sum_k [u_k^{(+)} e^{-i\omega_k^0 t + ikx} + u_k^{(-)} e^{+i\omega_k^0 t - ikx}], \quad (7.5)$$

$$\varphi(xt) = i \sum_k [u_k^{(+)} e^{-i\omega_k^0 t + ikx} - u_k^{(-)} e^{+i\omega_k^0 t - ikx}], \quad (7.6)$$

with the quadratic diffusive dispersion law

$$\omega_k^0 = -i\nu k^2. \quad (7.7)$$

Since u is real we have $(u_k^{(\pm)})^* = u_{-k}^{(\pm)}$, implying that φ is purely imaginary as discussed above. We also note that the solution (7.5), unlike the solution of the noiseless EW equation, includes both growing and decaying solutions. As discussed earlier this feature is consistent with the time reversal invariance in the stationary regime.

C. Nonlinear soliton modes

In order to treat the nonlinear aspects of the field equations we employ the same method as in the analysis of the noiseless Burgers equation and look for static solutions. Using the Galilean invariance propagating solutions are then easily generated by a transformation to a moving frame accompanied by a shift of u. The static case, $\partial u/\partial t = \partial \varphi/\partial t = 0$, corresponds to the action $S = -\int dx dt \mathcal{H}(u,\varphi)$ and the solutions are given by the stationary points of the Hamiltonian H. Multiplying the static field equations by $\nabla \varphi$ and ∇u, respectively, we obtain $\nabla\varphi\nabla^2\varphi + \nabla u\nabla^2 u = 0$, or by quadrature, imposing the boundary conditions of vanishing slope, $\nabla u = \nabla\varphi = 0$ for $x \to \pm L/2$, the slope condition

$$(\nabla\varphi)^2 + (\nabla u)^2 = 0. \quad (7.8)$$

Solving the slope condition we obtain

$$\nabla\varphi = i\mu\nabla u \quad (7.9)$$

parametrized by the parity index $\mu = \pm 1$, which inserted in the static field equations yields

$$\mu\nu\nabla^2 u + \lambda u\nabla u = 0. \quad (7.10)$$

This equation has the same form as the static limit of the noiseless Burgers equation (3.1) with damping $\mu\nu$, and we obtain the static solution (3.2) with ν replaced by $\mu\nu$, i.e., $\pm\nu$,

$$u(x) = u_+ \tanh\left[\frac{\mu\lambda u_+}{2\nu}(x - x_0)\right]. \quad (7.11)$$

The solution (7.11) has the form of a static, localized, symmetric soliton or kink with amplitude $2|u_+|$, center of mass x_0, width $2\nu/(\lambda|u_+|)$, approaching $\pm\mu|u_+|$ for $x \to \pm L/2$. In the limit of vanishing damping, $\nu \to 0$, the soliton becomes a sharp discontinuity or shock in the slope field. The static soliton connects two degenerate stationary states with slopes $\pm|u_+|$. However, unlike the sine-Gordon soliton [77,89], which is characterized by a topological quantum

number, or the φ^4 soliton [77,89], which connects the two degenerate ground states defined by the double-well potential, the Burgers soliton has an arbitrary amplitude $|u_+|$ corresponding to the infinitely degenerate stationary states. Furthermore, we note the interesting fact that unlike the case of the noiseless Burgers equation, where we only have a single *right hand* soliton mode, corresponding to $\mu = +1$, the broken reflection or parity symmetry is restored in the noisy case. The noise drives the interface into a stationary state and in the process excites both *right* and *left hand* solitons. This mechanism in the nonlinear case is equivalent to the excitation of both growing and decaying diffusive modes in the EW case. This ''doubling'' of soliton solutions in the driven case was also observed in Ref. [73].

The static soliton is a special configuration connecting stationary states with opposite slopes. However, since the underlying field equations are invariant under the Galilean symmetry group (6.7) and (6.8) it is an easy task to construct a propagating soliton solutions by means of a Galilean transformation. Similar to the discussion of the noiseless Burgers equation we obtain, introducing the boundary values $u_\pm$ for $x \to L/2$, the soliton condition (3.5), i.e.,

$$u_+ + u_- = -\frac{2v}{\lambda} \quad (7.12)$$

and the moving soliton solution

$$u(xt) = \frac{u_+ + u_-}{2} + \frac{u_+ - u_-}{2}$$
$$\times \tanh\left[\frac{\lambda}{4\nu}|u_+ - u_-|(x - vt - x_0)\right], \quad (7.13)$$

connecting stationary states with slopes u_+ and u_-; note that we have absorbed the parity index μ in the sign of $u_+ - u_-$. The soliton has center of mass x_0, propagates with velocity $v = -\lambda(u_+ + u_-)/2$, has the width $4\nu/(\lambda|u_+ - u_-|)$, and amplitude $|u_+ - u_-|$. In the limit of vanishing damping $\nu \to 0$, the inviscid limit, or large drive λ, the soliton becomes a shock wave, i.e., a discontinuity in the slope field; for finite damping the shock wave front is smoothed by dissipation. For vanishing amplitude, $u_+ - u_- \to 0$, the soliton merges continuously into the stationary state $u_+ = u_-$.

Finally, integrating the slope condition (7.8), which by inspection also holds for the propagating soliton, and using the invariance property (6.9) in order to set the integration constant equal to zero, we obtain

$$\varphi = i\mu u, \quad (7.14)$$

yielding the static and propagating soliton solutions for the associated noise field and in accordance with the saddle point regions in Fig. 7. In Fig. 8 we have depicted the static solitons and the associated smoothed cusps in the height field.

D. Multisoliton solutions: Boundary conditions

In addition to defining the path integral in a finite LT box, we must also specify appropriate boundary conditions for the slope and noise fields in accordance with the physical situation. For a growing interface it is convenient to assume a
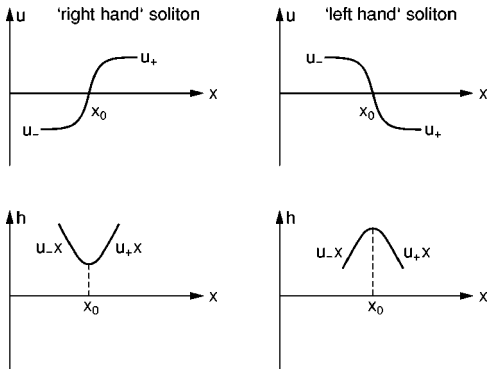
FIG. 8. In (a) and (b) we show the static *right* and *left hand* solitons and the smoothed static downward and upward cusps in the associated height field (arbitrary units).

horizontal interface at the boundaries equivalent to a vanishing slope field. This corresponds to a vanishing deterministic component in the current in Eq. (4.2) at the boundaries and implies that the interface is only driven by the noise. However, since the single soliton solution discussed above connects stationary states with different slopes corresponding to a nonvanishing current, we must pair at least two solitons of opposite parity in order to satisfy the boundary conditions. By inspection of the field equations (6.5) and (6.6) we note, however, that a nonoverlapping two-soliton configuration $u^{(1)}+u^{(2)}$ connected by a segment of constant slope is an approximate solution to the field equations and therefore corresponds to an extremum of the action in the path integral. The correction term is given by $\lambda(u^{(1)}\nabla u^{(2)}+u^{(2)}\nabla u^{(1)})$ whose contribution to the action we can ignore for nonoverlapping well-separated solitons. Furthermore, the argument can be generalized to a multisoliton configuration connected by segments of constant slope. It is a well-known feature of path integral instanton or solitons configurations that in order to obtain the correct asymptotic behavior one must sum over a gas of nonoverlapping instantons or solitons [89]. The situation is the same in the present somewhat more complicated context. In order to satisfy the boundary conditions of vanishing slope and to collect all the leading contributions in the asymptotic weak noise limit the structure of the path integral implies the formation of a dilute gas of nonoverlapping solitons. In Fig. 9 we have shown the case of two nonoverlapping soliton solutions.

VIII. DYNAMICS OF SOLITONS: PRINCIPLE OF LEAST ACTION

It is a fundamental aspect of the canonical form of the path integral for the noisy Burgers equation that it supports *a principle of least action* [86]. Therefore, unlike the noiseless
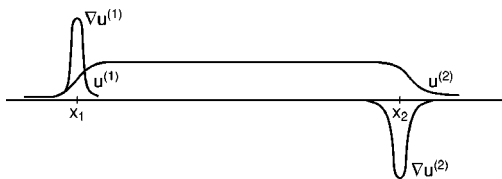

FIG. 9. We show the overlap between two well-separated solitons (arbitrary units).

Burgers equation where there is no underlying canonical structure, the asymptotic weak noise soliton and diffusive mode solutions are derived from a variational principle. Furthermore, we can associate an effective action, energy, and momentum with a particular phase space configuration or growth pattern.

The energy density ϵ is generally given by Eq. (5.5). Inserting the slope condition (7.8) valid for the soliton solutions, the harmonic part of ϵ cancels and the soliton solutions exclusively contribute to the growth term, i.e., $\epsilon = (\lambda/2)u^2\nabla\varphi$, or in terms of the momentum density (5.15), $\epsilon = (\lambda/2)ug$. Inserting the soliton constraint (7.9) the energy density also takes the form

$$\epsilon = (\lambda/2)i\mu u^2\nabla u. \qquad (8.1)$$

We note that the energy density is localized to the position of the soliton where u varies most rapidly. For the soliton energy we thus obtain by quadrature in terms of the boundary values $u_\pm$,

$$E = i\mu\frac{\lambda}{6}[u_+^3 - u_-^3]. \qquad (8.2)$$

In a similar manner the soliton momentum in Eqs. (5.14) and (5.15) is given by

$$P = i\mu\tfrac{1}{2}[u_+^2 - u_-^2] \qquad (8.3)$$

and finally from Eq. (5.4), using $\partial\varphi/\partial t = -v\nabla\varphi$ for the boosted static soliton, the soliton action

$$S = -T[Pv + E], \qquad (8.4)$$

which also follows from the Galilean invariance of S [86]. The surface contribution in Eq. (5.3) in terms of E and P has the form

$$S_B = -T[(\mu - 2)Pv + (2\mu - 3)E], \qquad (8.5)$$

and the functional G in Eq. (4.10) in the original MSR form is

$$G = \frac{\nu}{\Delta}T(1 - \mu)[Pv + 2E]. \qquad (8.6)$$

We notice that the asymmetry between *right* and *left hand* solitons is reflected in G. For the *right hand* soliton for $\mu = 1$, present in the noiseless case, G vanishes and the weight in $Z(\mu)$ in Eq. (4.9) is unity; the *left hand* soliton excited by the noise carries a nonvanishing G.

The purely imaginary character of E, P, and S is a feature of our choice of convention in establishing the canonical path integral in Eqs. (5.2)–(5.5). In order to exploit the formalism of analytical mechanics and the structure of the phase space Feynman path integral we have chosen the noise field φ in such a manner that u and φ appear as canonically conjugate variables satisfying the usual Poisson bracket (5.8). As discussed earlier this implies that φ has to be purely imaginary in the case of the weak noise saddle point solutions. The complex character of E and P is consistent with the relaxational and propagating aspects of the soliton modes as also observed in the Fokker-Planck description in paper II.

The main properties of E and P in the present dynamical context are that they serve as generators of translations in time and space, respectively [86]. The nonlinear energy-momentum relationship is characteristic of nonlinear soliton solutions [64] and is different from the simple E-P relationship encountered in the Lorentz invariant φ^4 or sine-Gordon equation [77]. We also note that the damping constant ν does not enter in the expressions for E and P, which only depend on the boundary values $u_{\pm}$ and the drive λ. In the weak noise limit the dynamics of soliton solutions is thus entirely decoupled from the dynamics of the linear diffusive modes.

Let us specifically consider a soliton configuration satisfying the boundary condition of left vanishing slope, i.e., $u_-=0$ for $x=-L/2$. The soliton condition (7.12) then implies the right boundary value $u_+=-2v/\lambda$, relating u_+ to the propagation velocity v. The soliton with positive parity, $u_+>0$, propagates left with negative velocity, whereas the soliton with opposite parity, $u_+<0$, propagates in the forward direction. For E and P we infer for both parities

$$E=\frac{4}{3}i\frac{|v|^3}{\lambda^2},\qquad(8.7)$$

$$P=-2i\frac{v|v|}{\lambda^2},\qquad(8.8)$$

$$S=\frac{2}{3}iT\frac{|v|^3}{\lambda^2}.\qquad(8.9)$$

The velocity $v=-\lambda u_+/2$ characterizes the kinematics of the soliton and is related to the amplitude whereas E and P determine the transformation properties. Eliminating the velocity we obtain the soliton dispersion law

$$E_P=i\lambda\frac{\sqrt{2}}{3}|P|^{3/2},\qquad(8.10)$$

where sgn(Im P)$=-$sgn v. We note that the nonlinear localized soliton excitation has a *qualitatively* different dispersion law from the linear extended diffusive mode dispersion law $\omega=-i\nu k^2$. They are both gapless modes but the exponents are different. The consequences of this aspect on the spectrum and scaling properties will be investigated later when we consider the fluctuations in more detail, but we already note here that the change in the exponent, which can be identified with the dynamic exponent z, is related to the different universality classes for the EW and Burgers cases.

Since the energy and momentum densities are localized at the soliton positions it follows that they are additive quantities for a multisoliton configuration and the general expressions (8.2), (8.3), and (8.4) allow us to evaluate the total energy, momentum, and action for an arbitrary configuration constructed from well-separated nonoverlapping solitons.

IX. A GROWING INTERFACE AS A DILUTE SOLITON GAS

We are now in position to present a coherent picture of the morphology of a statistically driven growing interface. In the weak noise limit $\Delta\rightarrow0$ *the principle of least action*, which operates in the present context, implies that the sta-

tionary points in (u,φ) phase space correspond to solitons and multisoliton configurations connected by segments of constant slope. In addition there will be superposed diffusive modes. In the nonlinear case the soliton configurations determine the dominant features of the growth morphology and will be considered here. The superposed diffusive modes will be discussed in the next section.

The weight of a particular soliton configuration in the path integral is given by the action that is an additive quantity for a dilute gas of solitons. The soliton configurations are assumed to be excited with respect to a stationary state of vanishing slope, i.e., a horizontal interface, and are furthermore determined by imposing periodic boundary conditions at $x=\pm L/2$; we remark that fixed boundary conditions are inconsistent with a soliton configuration moving across the system. We also note that unlike the transient properties of the noiseless Burgers equation, which are described by a gas of *right hand* solitons connected by ramps, corresponding to a transient height profile composed of smoothed cusps connected by convex parabolic segments as shown in Fig. 5, the stationary state of the noisy Burgers equation is characterized by a gas of both *right* and *left hand* solitons connected by pieces of constant slope. The noise thus radically changes the growth morphology of the Burgers equation. The noise stochastically modifies the transient regime by exciting solitons of both parities, which thus describe the morphology of the slope field in the stationary nonequilibrium state. The situation is similar to the case of the noise-driven damped sine-Gordon equation [90,91] where the noise also excites nonlinear soliton modes.

We now proceed to discuss the morphology of a growing interface in terms of solitons in the slope field u. The basic ''building block'' is the *static soliton* configuration given by Eq. (7.11). From Eqs. (8.2)–(8.4) it follows that this mode has vanishing momentum $P=0$, energy $E=i(\lambda/3)|u_+|^3$, and action $S=-i(\lambda/3)T|u_+|^3$, independent of its parity. By integration the height profile, $h=\int udx$, is given by

$$h(x)=\frac{2\nu}{\lambda}\frac{u_+}{|u_+|}\ln\cosh\left[\frac{\lambda u_+}{2\nu}(x-x_0)\right],\qquad(9.1)$$

corresponding to a downward and an upward pointing cusp smoothed by the damping constant ν. In the limit of vanishing ν the cusps become sharp. We also note that the static soliton does not satisfy the boundary conditions of vanishing slope. In Fig. 8 we have depicted the slope and height field in the two cases.

Boosting the static soliton in Eq. (7.11) we obtain a *single moving soliton* with velocity given by the soliton condition (7.12) and the profile (7.13). In the particular case of a soliton satisfying the left boundary condition $u_-=0$, we obtain from Eq. (7.13) the slope field

$$u(xt)=\frac{u_+}{2}\left\{1+\tanh\left[\frac{\lambda|u_+|}{4\nu}(x-vt-x_0)\right]\right\}\qquad(9.2)$$

and by integration the height profile

$$h(xt)=\frac{u_+}{2}x+\frac{2\nu}{\lambda}\frac{u_+}{|u_+|}\ln\cosh\left[\frac{\lambda|u_+|}{4\nu}(x-vt-x_0)\right]$$
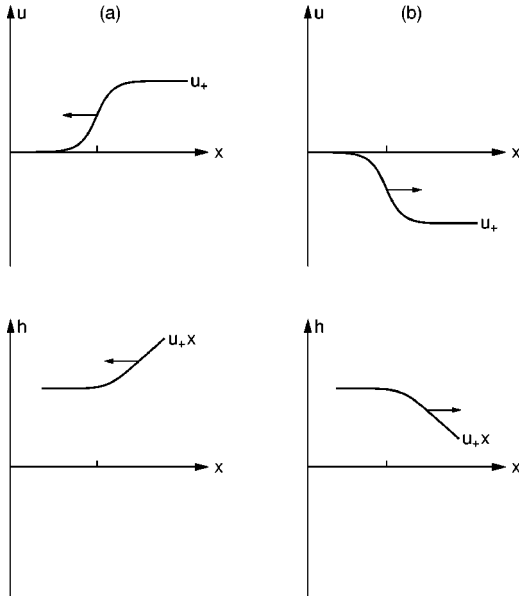
$$(9.3)$$

FIG. 10. In (a) and (b) we show right and left-moving solitons with vanishing amplitude at $x = -L/2$. The associated height profiles correspond to the bottom and top part of a step, respectively (arbitrary units).

with propagation velocity

$$v = -\frac{\lambda u_+}{2}. \tag{9.4}$$

The energy, momentum, and action are given by Eqs. (8.7) and (8.9). This mode corresponds to the bottom part of an ascending step or top part of a descending step in the height field propagating to the left or right, depending on the sign of u_+. The configurations are shown in Fig. 10.

In order to describe a *moving step* in the height profile we pair two well-separated non-overlapping solitons with equal amplitude and opposite parity. The soliton condition (7.12) then implies that they move in the same direction with the same velocity. In this case the slope and height fields have the form

$$u(xt) = \frac{u_+}{2}\left[\tanh\left(\frac{\lambda|u_+|}{4\nu}(x-vt-x_1)\right) - \tanh\left(\frac{\lambda|u_+|}{4\nu}\right.\right.$$
$$\left.\left. \times(x-vt-x_2)\right)\right], \tag{9.5}$$

$$h(xt) = \frac{2\nu}{\lambda}\frac{u_+}{|u_+|}\log\left[\frac{\cosh(\lambda|u_+|/4\nu)(x-vt-x_1)}{\cosh(\lambda|u_+|/4\nu)(x-vt-x_1)}\right], \tag{9.6}$$

$$v = -\frac{\lambda u_+}{2}. \tag{9.7}$$

We have here assumed $x_1 \ll x_2$ for the center of mass coordinates. This configuration corresponds to two comoving solitons moving with velocity $v = -\lambda u_+/2$ and is equivalent to a moving step in the height profile. The height of the step Δh is given by $\Delta h = u_+(x_2 - x_1)$. Imposing periodic boundary conditions for the slope field corresponding to a closed ring of length L, this two-soliton mode corresponds to a step
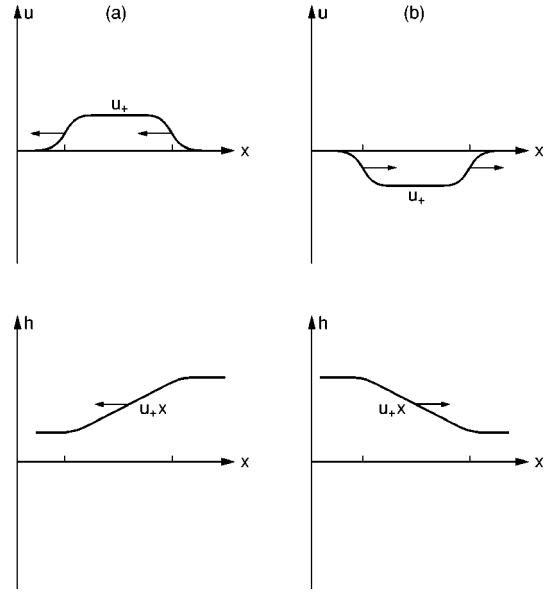

FIG. 11. In (a) and (b) we show two two-soliton configurations moving with opposite velocities corresponding to the left and right propagation of a step in the height profile (arbitrary units).

in h moving along the closed ring. At each revolution the height field thus increases by Δh and we have a simple growth situation. For well-separated solitons the energy, momentum, and action are additive and we obtain from Eqs. (8.2)–(8.4)

$$E_{\text{step}} = \frac{8}{3}i\frac{|v|^3}{\lambda^2}, \tag{9.8}$$

$$P_{\text{step}} = -4i\frac{v|v|}{\lambda^2}, \tag{9.9}$$

$$S_{\text{step}} = \frac{4}{3}iT\frac{|v|^3}{\lambda^2}. \tag{9.10}$$

It is also easily seen by inspection that for soliton configuration with vanishing slope field at the boundaries the surface contribution S_B in Eq. (5.3) vanishes. In Fig. 11 we have shown the configurations.

In a similar way we can construct a more faceted height profile in terms of a gas of appropriately paired solitons in the slope field u with the only requirement that (i) the solitons are well separated so that they constitute saddle point solutions and (ii) they satisfy periodic boundary conditions. For example, a *growing tip* or the *filling in* of an indentation is described by the three-soliton configurations. A *growing plateau* formed by two steps corresponds to a four-soliton configuration. We also notice that the two-soliton configurations corresponding to a moving step can be ''renormalized'' by the excitation of further two-soliton configurations corresponding to curvature of the step. In Fig. 12 we have depicted the above special configurations. In Fig. 13 we have shown a general profile.
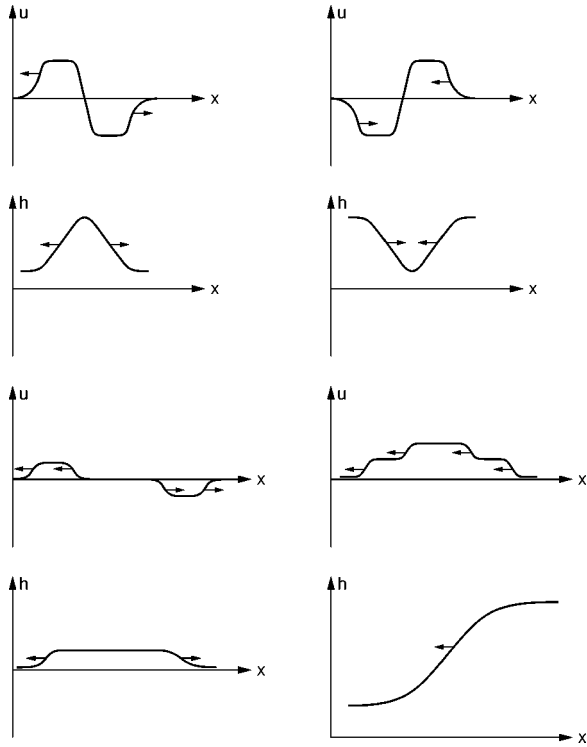
FIG. 12. We show the soliton configuration corresponding to (a) the growth of a tip, (b) the filling in of an indentation, (c) the growth of a plateau, and (d) the ''renormalization'' of a step (arbitrary units).

X. ''QUANTUM DESCRIPTION'' OF A GROWING INTERFACE: FLUCTUATIONS

In the previous section we demonstrated that the dominant morphology of a growing interface governed by the noisy Burgers equation in the weak noise limit can be described in terms of a dilute gas of propagating solitons. In the path integral the soliton contributions correspond to the stationary saddle points in the (u, φ) phase space determined by the principle of least action. By inspection of the one dimen-
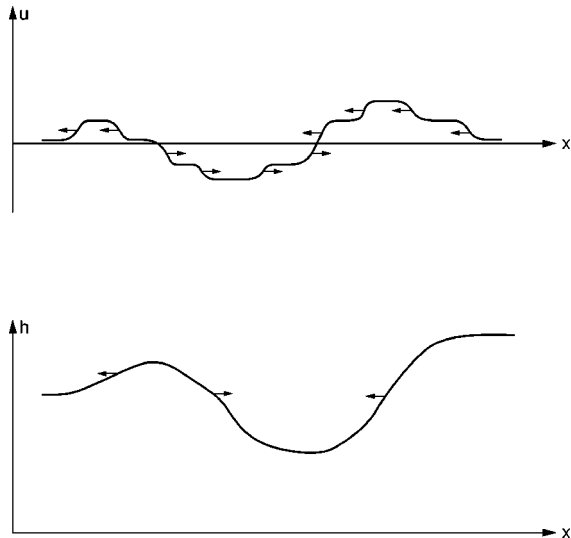
FIG. 13. The general growth of an interface in terms of a dilute gas of solitons (arbitrary units).

sional integral (6.1) and the saddle point result (6.2), it is clear that the soliton solution corresponds to the stationary point u_0 and the associated soliton action to $S(u_0)$. Consequently, we have not included the Gaussian fluctuations about the stationary point, yielding the multiplicative factor in Eq. (6.2) of order $\Delta^{1/2}$ and depending on the second order derivative $S''(u_0)$ evaluated at the stationary point, but have only taken into account the exponential contribution determined by the action. In the context of the path integral the Gaussian fluctuations about the stationary soliton correspond to the linear diffusive mode spectrum in the presence of the soliton configurations and remains to be discussed.

In order to proceed in the analysis of the path integral representation of the noisy Burgers equation we shall take a heuristic point of view and extract some information and physical insight by making use of the Feynman path integral structure of Z in Eqs. (5.2)–(5.5), deferring an analysis of the path integral *per se* to another context. The idea is to in a certain sense ''deconstruct'' the path integral and determine the form of the underlying ''quantum field theory'' leading to Z by the usual Feynman method [42]. Since the slope field u and the noise field φ in the path integral form a canonically conjugate pair with Poisson bracket (5.8), where u plays the role of a canonical ''momentum'' and φ a canonically conjugate ''coordinate,'' the first step is to introduce the ''quantum fields'' $\hat{u}$ and $\hat{\varphi}$, satisfying the canonical commutator

$$[\hat{u}(x), \hat{\varphi}(x')] = -i\frac{\Delta}{\nu}\delta(x-x'). \tag{10.1}$$

Here the ratio of the noise to the damping, Δ/ν, plays the role of an effective Planck constant just as in the path integral. We thus have an effective ''correspondence principle'' operating relating the ''classical'' Poisson bracket $\{A,B\}$ to the ''quantum'' commutator $[\hat{A},\hat{B}]$, according to the prescription $[\hat{A},\hat{B}] = -i(\Delta/\nu)\{A,B\}$. In a similar way the effective ''quantum Hamiltonian'' $\hat{H}$ is inferred from Eq. (5.9) [92],

$$\hat{H} = \int \left[-i\frac{\nu}{2}[(\nabla\hat{u})^2 + (\nabla\hat{\varphi})^2] + \frac{\lambda}{2}\hat{u}^2\nabla\hat{\varphi} \right] dx. \tag{10.2}$$

Whereas by construction the fields $\hat{u}$ and $\hat{\varphi}$ are Hermitian, the Hamiltonian $\hat{H}$ is in general a non-Hermitian operator. Expressing $\hat{H}$ in the form $\hat{H}_0 + \hat{H}_1$, it is composed of an anti-Hermitian harmonic component $\hat{H}_0$, governing the dynamics of the linear diffusive modes, and a nonlinear Hermitian component $\hat{H}_1$, describing the growth characterized by λ. In the Heisenberg picture $\hat{H}$ is the generator of time translations and we obtain the usual Heisenberg equations of motion [93]

$$\frac{\partial\hat{u}}{\partial t} = i\frac{\nu}{\Delta}[\hat{H},\hat{u}], \tag{10.3}$$

$$\frac{\partial\hat{\varphi}}{\partial t} = i\frac{\nu}{\Delta}[\hat{H},\hat{\varphi}], \tag{10.4}$$

yielding ''quantum field equations'' of the same form as the ''classical'' field equations (6.5) and (6.6) [94],

$$\frac{\partial \hat{u}}{\partial t} = -i \nu \nabla^2 \hat{\varphi} + \lambda \hat{u} \nabla \hat{u}, \qquad (10.5)$$

$$\frac{\partial \hat{\varphi}}{\partial t} = +i \nu \nabla^2 \hat{u} + \lambda \hat{u} \nabla \hat{\varphi}. \qquad (10.6)$$

In a similar way, the momentum operator $\hat{P}$, the generator of translation, is given by

$$\hat{P} = \int dx \hat{u} \nabla \hat{\varphi}, \qquad (10.7)$$

giving rise to the commutator relations

$$\nabla \hat{u} = i \frac{\nu}{\Delta} [\hat{P}, \hat{u}], \qquad (10.8)$$

$$\nabla \hat{\varphi} = i \frac{\nu}{\Delta} [\hat{P}, \hat{\varphi}]. \qquad (10.9)$$

Finally, the symmetry algebra in Sec. V also holds in the ''quantum case'' by simply replacing the Poisson brackets by commutators according to the above ''correspondence principle.''

The ''quantum field equations'' (10.5) and (10.6) together with the appropriate states of the Hamiltonian (10.2) are completely equivalent to the path integral and thus provide an alternative description of the noisy Burgers equation. The noise-induced fluctuations in the slope field, represented by the different configurations or paths in the path integral weighted by the ''classical'' action S, are replaced by ''quantum fluctuations'' in the underlying ''quantum field theory,'' resulting from the operator structure and the associated commutator algebra. We also note that the ''quantum description'' presented here is precisely the same as the one obtained in paper II based on the mapping of a solid-on-solid model to a continuum spin chain model in the quasiclassical limit.

A. The Edwards-Wilkinson equation

In order to demonstrate how the ''quantum scheme'' works it is instructive to evaluate the slope correlation function $\langle uu \rangle (k\omega)$ in Eq. (2.8) for the Edwards-Wilkinson equation. The dynamics of the EW case is governed by the unperturbed part of $\hat{H}$ in Eq. (10.2)

$$\hat{H}_0 = \int \left[-i \frac{\nu}{2} [(\nabla \hat{u})^2 + (\nabla \hat{\varphi})^2] \right] dx. \qquad (10.10)$$

Introducing the usual ''second quantization'' scheme [95] in terms of Bose annihilation and creation operators a_k and $a_k^\dagger$ satisfying the commutator algebra $[a_k, a_k^\dagger] = \delta_{kk'}$, we have for the slope and noise fields

$$\hat{u} = -i \sqrt{\frac{\Delta}{2\nu L}} \sum_k [e^{ikx} a_k - e^{-ikx} a_k^\dagger], \qquad (10.11)$$

$$\hat{\varphi} = \sqrt{\frac{\Delta}{2\nu L}} \sum_k [e^{ikx} a_k + e^{-ikx} a_k^\dagger], \qquad (10.12)$$

yielding the unperturbed Hamiltonian and associated diffusive dispersion law

$$\hat{H}_0 = \sum_k \frac{\Delta}{\nu} \omega_k^0 a_k^\dagger a_k, \qquad (10.13)$$

$$\omega_k^0 = -i \nu k^2. \qquad (10.14)$$

Noting that the particle vacuum state $|0\rangle$ corresponds to a stationary state with average vanishing slope $\langle 0|\hat{u}|0\rangle$, i.e., a horizontal interface, it is an easy task to evaluate the slope correlation function. Since the path integral defines a Bose time ordering [42] and using the time evolution operator, we have the identification

$$\langle u(x,t)u(0,0)\rangle = \langle 0|T\hat{u}(x,t)\hat{u}(0,0)|0\rangle$$
$$= \langle 0|\hat{u}(x)e^{-i\hat{H}_0|t|/(\Delta/\nu)}\hat{u}(0)|0\rangle. \qquad (10.15)$$

Using that a_k evolves in time according to

$$a_k(t) = a_k(0)\exp\left(-i\frac{\Delta}{\nu}\omega_k^0 t\right) \qquad (10.16)$$

we then obtain in Fourier space

$$\langle u(k,\omega)u(-k,-\omega)\rangle = i\frac{\Delta}{2\nu}\left[\frac{\langle 0|a_k a_k^\dagger|0\rangle}{\omega_k^0 - \omega} + \frac{\langle 0|a_k a_k^\dagger|0\rangle}{\omega_k^0 + \omega}\right] \qquad (10.17)$$

or in reduced form in complete agreement with Eq. (2.8),

$$\langle u(k,\omega)u(-k,-\omega)\rangle = \frac{\Delta k^2}{\omega^2 - (\omega_k^0)^2}. \qquad (10.18)$$

This simple calculation demonstrates how the ''quantum fluctuations'' as expressed by the commutator algebra and the effective Planck constant Δ/ν combine to produce the factor Δ in the correlation function, which ''classically'' in terms of the EW Langevin equation originates from averaging over the noise $\nabla \eta$.

B. The ''quantum soliton''

In the nonlinear case the ''quantum dynamics'' is governed by $\hat{H} = \hat{H}_0 + \hat{H}_1$ in Eq. (10.2). Introducing the Bose field $\hat{\psi}$ in configuration space,

$$\hat{\psi}(x) = (1/\sqrt{L}) \sum_k a_k \exp(ikx) \qquad (10.19)$$

the Hamiltonian takes the form

$$\hat{H} = (-i)(\Delta/\nu) \int dx |\nabla \hat{\psi}^\dagger|^2 - \lambda(\Delta/2\nu)^{3/2}$$
$$\times \int dx (\hat{\psi}^\dagger - \hat{\psi})^2 \nabla(\hat{\psi}^\dagger + \hat{\psi}) \qquad (10.20)$$

describing the many-body interaction between the linear diffusive modes governed by the first term $\hat{H}_0$.

Imposing the constraint of a horizontal interface we obtain $\langle \hat{u} \rangle \propto \langle \hat{\psi} - \hat{\psi}^{\dagger} \rangle = 0$, which implies that $\langle \hat{\psi} \rangle = \langle \hat{\psi}^{\dagger} \rangle$. Since the interaction term $\hat{H}_1$ does not conserve the number of particles, this constraint can only be satisfied for nonvanishing $\langle \hat{\psi} \rangle$ if the diffusive modes condense into a *coherent condensate* so that $\langle \hat{\psi} \rangle = \langle \hat{\psi}^{\dagger} \rangle \neq 0$. The resulting macroscopic wave function or condensate corresponds to the ''classical'' soliton mode discussed in the previous sections. The situation is quite similar to the phenomenological theory for superfluid helium based on a condensate wave function. The condensate has two components, $\langle \hat{\psi} \rangle$ and $\langle \hat{\psi}^{\dagger} \rangle$ or $\langle \hat{u} \rangle = u$ and $\langle \hat{\varphi} \rangle = \varphi$, and satisfies the coupled field equations (6.5) and (6.6), obtained from the ''quantum field equations'' (10.5) and (10.6) by ignoring ''quantum fluctuations'' and replacing the terms, $\lambda \hat{u} \nabla \hat{u}$ and $\lambda \hat{u} \nabla \hat{\varphi}$, by their average values, $\lambda u \nabla u$ and $\lambda u \nabla \varphi$. We can thus regard Eqs. (6.5) and (6.6) as two coupled Gross-Pitaevsky-type equations for the condensate wave function or soliton mode [96].

The ''classical'' soliton is localized in space and carries energy and momentum, depending on the boundary conditions according to the expressions (8.2) and (8.3). Subject to ''quantization'' this mode becomes a *bona fide* ''quantum mechanical'' quasiparticle with the same energy and momentum. Notice, however, that the ''quantum soliton'' is delocalized owing to the ''uncertainty principle,'' which implies that $\Delta x_0 \Delta P \sim \Delta / \nu$; here Δx_0 is the uncertainty in the center of mass position for the soliton and ΔP the uncertainty in its momentum. For a ''quantum soliton'' with well-defined momentum P and energy E we can in the usual way associate a wave number K and a frequency Ω, according to the ''de Broglie'' relations, $P = (\Delta / \nu) K$ and $E = (\Delta / \nu) \Omega$, and describe the quasiparticle by means of the wave function $\Psi \propto \exp[-i\Omega t + iKx]$. Considering in particular a pair of ''quantum solitons,'' describing a propagating step in the height profile with energy and momentum given by Eqs. (9.8) and (9.9), we obtain

$$\Omega_{\text{step}} = i \left(\frac{\nu}{\Delta} \right) \frac{8}{3} \frac{|v|^3}{\lambda^2}, \qquad (10.21)$$

$$K_{\text{step}} = -i \left(\frac{\nu}{\Delta} \right) 4 \frac{v|v|}{\lambda^2}. \qquad (10.22)$$

and the wave function takes the form

$$\Psi \propto \exp[-i\Omega t + iKx] = \exp[\text{const} \times (x - v_{\text{ph}} t)], \qquad (10.23)$$

corresponding to a propagation with phase velocity $v_{\text{ph}} = (2/3)v$. Noting, however, that the appropriate wave function for a localized soliton is the wave packet construction,

$$\Psi_{\text{WP}} \propto \sum_K A_K \exp[-i\Omega t + iKx] \qquad (10.24)$$

obtained from a superposition of plane waves, we obtain the group velocity

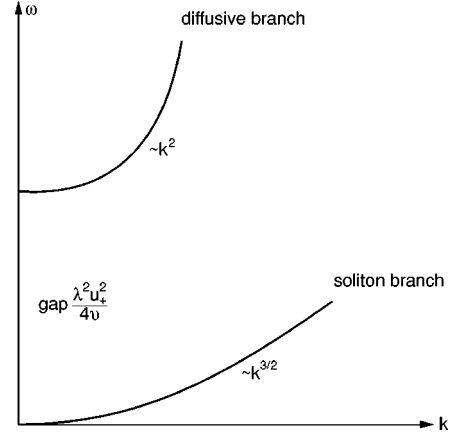$$v_g = \frac{d\Omega}{dK} = \frac{d\Omega/dv}{dK/dv} = v. \qquad (10.25)$$

FIG. 14. We depict the quadratic diffusive dispersion law with gap $\lambda^2 u_+^2 / 4\nu$ and the gapless soliton dispersion law with fractional exponent 3/2 (arbitrary units).

This shows that the quasiparticle wave packet propagates with the same velocity as the ''classical'' soliton in complete accordance with ''the correspondence principle.'' Whereas the propagation velocity v determines the kinetics of the ''classical'' soliton, the energy and momentum are the fundamental characteristics in the ''quantum'' case; the velocity v becomes the group velocity of the wave packet. We also notice from the wave packet form in Eq. (10.24) that the ''quantum soliton'' corresponds to a propagating mode. Finally, eliminating the velocity from Eqs. (10.21) and (10.22) we derive the ''quantum soliton'' dispersion law

$$\Omega_K = i\lambda \frac{1}{3} \left(\frac{\Delta}{\nu} \right)^{1/2} |K|^{3/2}, \qquad (10.26)$$

where $\text{sgn}(\text{Im } K) = -\text{sgn } v$.

C. ''Quantum fluctuations''

The final issue to consider in the qualitative ''quantization'' of the soliton system is the role of ''quantum fluctuations'' in the presence of a ''quantum soliton.'' This problem is treated here by expanding the fields $\hat{u}$ and $\hat{\varphi}$ about a soliton or condensate configuration (u_0, φ_0). Inserting $\hat{u} = u_0 + \delta \hat{u}$ and $\hat{\varphi} = \varphi_0 + \delta \hat{\varphi}$ in Eqs. (10.5) and (10.6) we obtain to linear order two coupled equations for the ''quantum fluctuations'' $\delta \hat{u}$ and $\delta \hat{\varphi}$,

$$\frac{\partial \delta \hat{u}}{\partial t} = -i\nu \nabla^2 \delta \hat{\varphi} + \lambda u_0 \nabla \delta \hat{u} + \lambda (\nabla u_0) \delta \hat{\varphi}, \quad (10.27)$$

$$\frac{\partial \delta \hat{\varphi}}{\partial t} = +i\nu \nabla^2 \delta \hat{u} + \lambda u_0 \nabla \delta \hat{\varphi} + \lambda (\nabla \varphi_0) \delta \hat{u}. \quad (10.28)$$

These equations have the same form as the ones obtained by expanding in the Gaussian fluctuations about the stationary soliton solution in the path integral.

The equations of motion (10.27) and (10.28) describe the interaction of the linear ''quantum diffusive modes'' $(\delta \hat{u}, \delta \hat{\varphi})$ with the soliton configuration (u_0, φ_0) and constitute a generalization to the noisy case of the linear stability

equation in the analysis in paper I; the soliton again acts like a potential giving rise to phase shift effects and a gap in the diffusive spectrum.

As in the noiseless case, Eqs. (10.27) and (10.28) admit an analytical solution. Since the equations are Galilean invariant we need only consider the case of a static soliton. First noting that the soliton solution according to Eq. (7.9) is confined to the diagonal lines $\nabla \varphi_0 = i \mu \nabla u_0$, $\mu = \pm 1$, in phase space, the fluctuations $\delta \hat{u}$, $\delta \hat{\varphi}$ are disentangled by transforming to ''normal coordinates'' along and perpendicular to the ''soliton lines.'' Thus (1) introducing ''normal coordinates'' $\delta \hat{X} = \delta \hat{u} + i \delta \hat{\varphi}$, $\delta \hat{Y} = \delta \hat{u} - i \delta \hat{\varphi}$, (2) inserting the static solution (7.11), $u_0 = \mu |u_+| \tanh(k_s x)$, $\nabla u_0 = \mu |u_+| k_s \cosh^{-2}(k_s x)$, $\nabla \varphi_0 = \mu \nabla u_0$, and (3) performing the scaling transformations $\delta \hat{X} \to h \delta \hat{X}$, $\delta \hat{Y} \to h^{-1} \delta \hat{X}$, where $h = \cosh^{\mu}(k_s x)$, in order to absorb the linear terms in ∇, i.e.,

$$\delta \hat{X} = h^{-1}(\delta \hat{u} + i \delta \hat{\varphi}), \qquad (10.29)$$

$$\delta \hat{Y} = h(\delta \hat{u} - i \delta \hat{\varphi}), \qquad (10.30)$$

$$h = \cosh^{\mu}(k_s x), \qquad (10.31)$$

$$k_s = \frac{\lambda |u_+|}{2 \nu} \qquad (10.32)$$

we arrive at the effectively decoupled equations for the ''normal coordinates''

$$\frac{\partial \delta \hat{X}}{\partial t} = + D \delta \hat{X} + (\mu - 1) \nu k_s^2 \delta \hat{Y}, \qquad (10.33)$$

$$\frac{\partial \delta \hat{Y}}{\partial t} = - D \delta \hat{X} + (\mu + 1) \nu k_s^2 \delta \hat{Y}. \qquad (10.34)$$

Here the Schrödinger operator D has the same form as the stability matrix for the noiseless Burgers equation in paper I or the sine Gordon equation [77,80],

$$D = - \nu \nabla^2 + \nu k_s^2 \left[1 - \frac{2}{\cosh^2(k_s x)} \right]. \qquad (10.35)$$

The wave number $k_s = \lambda |u_+| / 2 \nu$ introduced in Sec. III depending on the soliton amplitude sets the inverse length scale. We also note that Eqs. (10.33) and (10.34) reduce to the linear case for $\lambda = 0$ since $D \to - \nu \nabla^2$ and $h \to 1$.

Since the Bargman potential $\cosh^{-2}(k_s x)$ admits an exact solution the spectrum of D defined by the eigenvalue equation $D \Psi_n = i \omega_n \Psi_n$ is well known [97] and is discussed in paper I. It is composed of a zero-eigenvalue localized bound state mode and a band of phase shifted scattering modes,

$$\Psi_0 \propto \frac{1}{\cosh(k_s x)}, \qquad (10.36)$$

$$\omega_0 = 0, \qquad (10.37)$$

$$\Psi_k \propto \exp(ikx) \frac{k + i k_s \tanh(k_s x)}{k - i k_s}, \qquad (10.38)$$

$$\omega_k = - i \nu (k^2 + k_s^2). \qquad (10.39)$$

Expanding $\delta \hat{X}$ and $\delta \hat{Y}$ on a set of eigenstates Ψ_n,

$$\delta \hat{X} = \sum_n \hat{a}_n \Psi_n, \qquad (10.40)$$

$$\delta \hat{Y} = \sum_n \hat{b}_n \Psi_n \qquad (10.41)$$

we finally obtain equations of motion for the expansion coefficients $\hat{a}_n$ and $\hat{b}_n$,

$$\frac{d \hat{a}_n}{dt} = + i \omega_n \hat{a}_n + (\mu - 1) \nu k_s^2 \hat{b}_n, \qquad (10.42)$$

$$\frac{d \hat{b}_n}{dt} = - i \omega_n \hat{b}_n + (\mu + 1) \nu k_s^2 \hat{a}_n, \qquad (10.43)$$

which we proceed to discuss.

1. The translation modes

The zero-frequency mode of the Schrödinger operator D in Eq. (10.35) is associated with the translation and boosting of the static soliton profile (u_0, φ_0). This is seen in the following way. Since the ''quantum field equations'' (10.5) and (10.6) have the same form as the ''classical'' field equations (6.5) and (6.6) they are equally satisfied by a soliton solution. Consequently, a variation of the static soliton profile, $(\delta u_0, \delta \varphi_0)$, is a solution of the linearized equations (10.27) and (10.28) or (10.33) and (10.34) in the static case, corresponding to the bound state $\Psi_0 = 0$. Furthermore, since the soliton depends parametrically on the center of mass position x_0 it follows that the fluctuations $(\delta u_0, \delta \varphi_0)$ are proportional to the derivatives $(\nabla u_0, \nabla \varphi_0)$ with respect to x_0, corresponding to a displacement of the soliton position. This mode is thus a translation or Goldstone mode associated with the broken translational symmetry and is a well-known feature of symmetry breaking ''excitations''; a Goldstone mode is excited in order to restore the broken symmetry [36]. A similar translation mode was also encountered in our discussion of the noiseless case in paper I.

Focusing, for example, on the *right hand* static soliton for $\mu = +1$ and solving Eqs. (10.42) and (10.43) for $n = 0$ we have the expansion coefficients

$$\hat{a}_0 = \text{const}, \qquad (10.44)$$

$$\hat{b}_0 = \hat{b}_0^0 + 2 \nu k_s^2 \hat{a}_0 t, \qquad (10.45)$$

where $\hat{b}_0^0$ is the initial value for $t = 0$, and we obtain, using that $\nabla u_0 \propto \cosh^{-2}(k_s x)$, the fluctuation mode

$$\delta \hat{u} = \hat{a}_0 + (\hat{b}_0^0 + \hat{a}_0 \nu k_s^2 t) \nabla u_0. \qquad (10.46)$$

For $\hat{a}_0 = 0$ this mode corresponds to an infinitesimal translation $\delta x_0 = \hat{b}_0^0$ of the soliton, i.e., a change of the center of

mass coordinate; for $\hat{a}_0 \neq 0$ the mode is equivalent to a boost of the soliton to a small velocity $\propto \nu k_s$. A similar discussion applies to $\delta \hat{\varphi}$.

2. The diffusive scattering modes

The band of diffusive scattering modes is also easily discussed. From the equations of motion for the expansion coefficients in Eqs. (10.42) and (10.43) and again considering a *right hand* static soliton for $\mu = 1$ we obtain for $n = k$ the solution

$$\hat{a}_k = \hat{a}_k^0 e^{i\omega_k t}, \tag{10.47}$$

$$\hat{b}_k = \hat{a}_k^0 \left[\frac{k_s^2}{k^2 + k_s^2} \right] e^{i\omega_k t} + \left\{ \hat{b}_k^0 - \hat{a}_k^0 \left[\frac{k_s^2}{k^2 + k_s^2} \right] \right\} e^{-i\omega_k t}. \tag{10.48}$$

Here a_k^0 and b_k^0 are the initial values and the spectrum ω_k given by Eq. (10.39). For the fluctuation $\delta \hat{u}$ we then have

$$\delta \hat{u} = \sum_k \left[\hat{a}_k \cosh(k_s x) + \hat{b}_k \cosh^{-1}(k_s x) \right] \Psi_k, \tag{10.49}$$

where Ψ_k is given by Eq. (10.38). We note that $\delta \hat{u}$ in the soliton case again is composed of both positive and negative frequency parts, $\exp[\nu(k^2 + k_s^2)t]$ and $\exp[-\nu(k^2 + k_s^2)t]$, corresponding to growing and decaying modes in the stationary driven regime, exhibiting a gap νk_s^2 in the spectrum. In the linear EW case $k_s = 0$ and $\delta \hat{u}$ assumes the form in Eq. (10.11). We shall not dwell here on the somewhat complicated x dependence but only observe that the main effect of the soliton on the diffusive modes apart from phase shift effects and spatial modulations is to lift the spectrum and create a gap $\nu k_s^2 = \lambda^2 |u_+|^2 / 4\nu$, depending on the soliton amplitude u_+, the coupling λ, and the frequency ν.

D. Many-body description of a growing interface

The above analysis of the "quantum solitons" and the "quantum diffusive modes" allows a heuristic qualitative discussion of a growing interface. The stochastic dynamics of the noisy Burgers equation (4.1) in the stationary regime can be rigorously interpreted in terms of a dilute Landau-type "quantum" quasiparticle gas composed of elementary excitations of two types: "Quantum solitons" and "quantum diffusive modes." The "quantum mechanics" being equivalent to a Master equation description [56,57] is basically relaxational, corresponding to a complex Hamiltonian.

The elementary excitations fall into two classes: linear diffusive modes and nonlinear soliton modes. (1) The linear diffusive modes are associated with the damping term in the Burgers equation or, equivalently, the harmonic anti-Hermitian part in the Hamiltonian. These modes account for the relaxational aspects of the interface and are characterized by the dispersion law (10.39), i.e.,

$$\omega_k = -i\nu(k^2 + k_s^2) \tag{10.50}$$

with a gap νk_s^2. As in our discussion of the noiseless case in I the gap can be associated with a nonvanishing current towards the center of the soliton where the damping in enhanced. (2) The nonlinear soliton modes are related to the

nonlinear growth term in the Burgers equation or, equivalently, the nonlinear Hermitian part of the Hamiltonian. For a pair of solitons representing a growing step the dispersion law is given by Eq. (10.26), i.e.,

$$\Omega_K = i\frac{\lambda}{3} \left(\frac{\Delta}{\nu} \right)^{1/2} |K|^{3/2}. \tag{10.51}$$

The mode is gapless and characterized by the fractional exponent 3/2. The soliton mode accounts for the growth aspects of the driven interface. For $\nu \to \infty$ the linear damping dominates the growth and $\Omega_K \to 0$, also for $\lambda \to 0$ we attain the linear EW case and $\Omega_K \to 0$; finally for $\Delta \to 0$ the stochastic aspects are quenched, solitons (and diffusive modes) are not kinetically or stochastically excited, and $\Omega_K \to 0$.

In the linear EW case the fluctuating interface is in equilibrium and here described as a noninteracting gas of linear gapless diffusive modes. The statistical fluctuations appear as "quantum fluctuations" of the quasiparticle modes. Since the dispersion is quadratic we can also envisage the EW case as a "quantum" gas of free particles with imaginary mass.

In the nonlinear Burgers case the "quantum soliton" emerges as a new additional quasiparticle, corresponding to the faceted genuine growth of an interface. The linear modes become subdominant in the sense that they develop a gap in the spectrum and correspond to superposed damped "ripple modes" on the soliton configurations. The diffusive modes extend over the whole configuration and are phase shifted due to reflectionless scattering against the solitons as in the noiseless case. A scattering analysis also, in accordance with Levinson's theorem, shows the diffusive spectrum is depleted by a number of states, corresponding to the translation modes of the solitons.

Before turning to the heuristic scaling analysis in the next section, we wish to add a few more remarks concerning the structure of a field theoretic or many-body description of the noisy Burgers equation. There are basically two equivalent modes of approach: (1) A direct evaluation of the path integral in the weak noise limit in a saddle point approximation for a dilute soliton gas, including the diffusive modes, corresponding to Gaussian fluctuations about the saddle points, and summing over periodic orbits in order to include secular effects or (2) a construction of the equivalent "quantum many-body theory" on the basis of the "quantum representation" of the path integral. Including the soliton modes as space- and time-dependent *condensate* configurations, as mentioned in Sec. X B, the many-body approach is similar to the microscopic theory of interacting bosons [95,98,99] with anomalous propagators, etc. There are, however, some notable differences. In the case of interacting bosons the uniform condensate acts as a particle reservoir and changes the free boson dispersion law $\omega \propto p^2$ to a linear acoustic phonon branch $\omega \propto p$. In the present case, the condensate corresponding to a soliton or gas of solitons is nonuniform and time dependent, governed by the "classical" field equations. The free diffusive modes with dispersion $\omega \propto p^2$ develop a gap depending on the soliton amplitude or velocity, whereas the soliton or condensate mode emerges as a new quasiparticle with dispersion $\omega \propto p^{3/2}$.

XI. SCALING AND UNIVERSALITY CLASSES

In addition to providing a many-body description of a growing interface in terms of a Landau-type quantum quasi-particle gas of propagating ''quantum solitons'' and damped ''quantum diffusive modes,'' the path integral formulation also offers as a by-product some insight into the scaling properties, i.e., the behavior of the interface in the limit of large distances and long times.

We shall here focus on the scaling properties of the slope correlation function summarized in the dynamical scaling form (1.7), i.e., assuming $t = 0$ in the stationary regime,

$$\langle u(x,t)u(0,0)\rangle = |x|^{2(\zeta-1)}f(|t|/|x|^z). \quad (11.1)$$

The scaling issue is then to determine the roughness or wandering exponent ζ, the dynamic exponent z, and the scaling function $f(w)$.

In the EW case the scaling function f is given by Eq. (2.9), i.e.,

$$f(w) = (\Delta/2\nu)(4\pi\nu)^{-1/2}w^{-1/2}\exp[-1/4\nu w], \quad (11.2)$$

implying the exponents $(\zeta,z) = (1/2,2)$. In the Burgers case Galilean invariance together with the fixed point structure lead to the scaling law (1.10), i.e., $\zeta + z = 2$, which together with the stationary distribution (1.11), an effective fluctuation-dissipation theorem, yields the exponents $(\zeta,z) = (1/2,3/2)$.

According to the path integral formulation in Secs. IV and V, using Eqs. (4.7) and (5.2) the slope correlation function is given by

$$\langle u(x,t)u(0,0)\rangle = Z(0)^{-1}\int\prod_{xt}du\,d\varphi$$
$$\times\exp\left[i\frac{\nu}{\Delta}S\right]u(xt)u(00), \quad (11.3)$$

or in terms of the underlying ''quantum field theory,'' noting that the path integral by construction defines time-ordered products [42],

$$\langle u(x,t)u(0,0)\rangle = \langle 0|T\hat{u}(x,t)\hat{u}(0,0)|0\rangle. \quad (11.4)$$

Here $|0\rangle$ denotes the appropriate stationary state for the system.

In order to elucidate the structure of Eq. (11.4) we construct a spectral representation by (i) displacing the slope field $\hat{u}(x,t)$ to the origin in space and time by means of the Hamiltonian $\hat{H}$ and the momentum $\hat{P}$, using the integrated form of the commutator relations (10.3) and (10.8), and (ii) inserting intermediate eigenstates $|P\rangle$ with momentum P and energy E_P. The first step implies the relation

$$\hat{u}(x,t) = \exp\left[i\frac{\nu}{\Delta}(\hat{P}x + \hat{H}t)\right]\hat{u}(0,0)\exp\left[-i\frac{\nu}{\Delta}(\hat{P}x + \hat{H}t)\right]; \quad (11.5)$$

secondly, inserting intermediate states, using $\hat{H}|P\rangle = E_P|P\rangle$ and $\hat{P}|P\rangle = P|P\rangle$, introducing the frequency and wave number (Ω_K, K) associated with the elementary excitations or quasiparticles, and lumping the matrix elements in an effective form factor, $G(K) = \langle 0|\hat{u}|K\rangle\langle K|\hat{u}|0\rangle$, we arrive at the spectral representation

$$\langle u(x,t)u(0,0)\rangle = \int\frac{dK}{2\pi}G(K)\exp[-i(\Omega_K|t| - K|x|)]. \quad (11.6)$$

The time ordering in Eq. (11.4) together with parity invariance, $x \to -x$, imply evenness in the dependence on x and t; also $G(K)$ must be even in K.

The spectral form (11.6) is only schematic. For a multi-soliton diffusive mode intermediate eigenstate $|\{K_i\},\{k_j\}\rangle$, where K_i and k_j denote the soliton and diffusive mode wave numbers, respectively, with total wave number $K = \Sigma_i K_i + \Sigma_j k_j$ and total frequency $\Omega = \Sigma_i\Omega_{K_i} + \Sigma_j\omega_{k_j}$, we have strictly speaking the spectral form, say for $t > 0$,

$$\langle u(x,t)u(0,0)\rangle = \int\prod_{ij}dK_i dk_j G(\{K_i\},\{k_j\})$$
$$\times e^{-i[(\Sigma_i K_i + \Sigma_j k_j)x + (\Sigma_i\Omega_{K_i} + \Sigma_j\omega_{k_j})t]}. \quad (11.7)$$

Since the solitons are transparent with respect to the diffusive modes as discussed in Sec. X C, the operator $\hat{u}$ only excites a single mode k extending across the system, i.e., $G(\{K_i\},\{k_j\}) \sim G(\{K_i\},k)$, and assuming furthermore that $G(\{K_i\},k)$ factorizes approximately in accordance with the dilute soliton gas picture, $G(\{K_i\},k) \sim G_D(k)\Pi_i G_S(K_i)$, summing over the solitons we obtain

$$\langle u(x,t)u(0,0)\rangle \sim \frac{\int dk\, G_D(k)e^{-ikx-i\omega t}}{1 - \int dK\, G_S(K)e^{-iKx-i\Omega t}}. \quad (11.8)$$

The expression (11.8) is clearly not correct in detail since we have not solved the many-body problem but only made some plausible assumptions concerning the form factor G. Nevertheless, from the point of view of discussing the scaling properties Eq. (11.8) has the required structure and serves our purpose. In the EW case $G_S(K) = 0$ and Eq. (11.8) reduces to the scaling form (11.2). In the Burgers case $\Omega \propto i|K|^{3/2}$, i.e., $\exp(-i\Omega t) = \exp(+\text{const}\times|K|^{3/2}t)$, and the denominator $[1 - \int dK G_S(K)e^{-iKx-i\Omega t}]^{-1} \sim \int dK G_S(K)e^{-iKx-i\Omega t}$ controls the scaling behavior. In both cases we can use the simplified general spectral form (11.6).

For the purpose of a discussion of the scaling properties we first consider a general quasi-particle dispersion law with a gap $\widetilde{\Delta}$, stiffness constant A, and exponent β,

$$\Omega_K = \widetilde{\Delta} + A|K|^\beta. \quad (11.9)$$

For large distances $|x| \gg a$, where a is a microscopic length defining the UV cutoff $K \sim 1/a$ implied in Eq. (11.6), the spectral representation samples the small wave number regions $K \ll 1/a$. Assuming that the form factor is regular for small K, i.e., $G(K) \sim G(0) + (1/2)K^2 G''(0)$, and rescaling

K, $Kx \rightarrow K$, we obtain, inserting the general dispersion (11.9), the spectral representation in a more appropriate scaling form

$$\langle u(x,t)u(0,0)\rangle \sim G(0)e^{-i\widetilde{\Delta}t}x^{-1}\int \frac{dK}{2\pi}e^{-iA|K|^{\beta}|t|/|x|^{\beta}-iK}.$$

(11.10)

We emphasize again that the spectral representation (11.10) can only be considered as a heuristic expression since we have not here carried out a detailed analysis of the non-Hermitian non-Lagrangian field theory underlying the path integral. Nevertheless, we believe that we can already draw some interesting general consequences concerning the scaling properties of a growing interface.

First we observe that in the presence of a gap $\widetilde{\Delta}\neq0$ there is no scaling behavior. For the diffusive mode in the presence of a soliton the spectrum is, for example, given by Eq. (10.50) with a gap $\widetilde{\Delta}=-i\nu k_s^2$, implying an exponential falloff with t in Eq. (11.10). Consequently, only gapless excitations for $\widetilde{\Delta}=0$ contribute to the scaling behavior. The gapless excitations are associated with the so-called zero-temperature fixed point behavior of the ''quantum field theory'' and determine the scaling properties.

Furthermore, comparing the spectral form (11.10) for $\widetilde{\Delta}=0$ with the scaling form (11.1) we immediately identify the roughness exponent $\zeta=1/2$ and the dynamic exponent $z=\beta$. We also note that whereas the exponent $\zeta=1/2$ essentially follows from a simple *regularity property* of the form factor $G(K)$ with leading term $G(0)$ for small K, the exponent z is tied to the exponent β in the quasiparticle dispersion law.

In the linear EW case the diffusive gapless modes with dispersion law (10.14), i.e., $\omega_k=-i\nu k^2$, exhaust the spectrum and we obtain $\beta=z=2$, corresponding to the EW universality class in Table I. Also the spectral form yields the scaling function (11.2) with the identification $G(0)=\Delta/2\nu$.

In the nonlinear Burgers-KPZ case the soliton modes with gapless dispersion (10.51), i.e., $\Omega_K\propto(\Delta/\nu)^{1/2}\lambda|K|^{3/2}$, exhaust the bottom of the spectrum and yield the exponent $\beta=z=3/2$, corresponding to the Burgers-KPZ universality class in Table I; the linear diffusive modes develop a gap according to Eq. (10.50), become subdominant, and do not contribute to the scaling behavior.

The above discussion thus provides a *dynamical interpretation* of the scaling properties, exponents, and universality classes. The universality class is determined by the lowest-lying gapless excitation. The spectral form also elucidates the *robustness* of the roughness exponent ζ, which is the same for both universality classes. In the case of the stationary equal-time fluctuations we set $t=0$ in Eq. (11.8) and the resulting scaling form yielding ζ does *not* depend on the specific quasiparticle dispersion law; this argument is equivalent to the effective fluctuation-dissipation theorem yielding the stationary distribution (1.11) *independent* of the nonlinear drive λ.

The spectral form (11.10) also provides an expression for the scaling function $f(w)$. We obtain comparing Eq. (11.8) with Eq. (11.1)

$$f(w)=G(0)\int \frac{dK}{2\pi}e^{-i(A|K|^z w+K)}.$$

(11.11)

The above expression is at best heuristic but we do notice that it has the correct limiting behavior, i.e., $f(w)\sim$ const for $w\rightarrow0$ and $f(w)\sim w^{-1/z}$ for $w\rightarrow\infty$.

The scaling function $f(w)$ describing the behavior of the strong coupling fixed point has been accessed both numerically [51–53] and by means of an analytical mode coupling approach [54,55], based on a self-consistent one-loop calculation, i.e., to first order in λ, and ignoring vertex corrections; we note that it has been shown in Ref. [55] that the vertex corrections up to second loop order are nonsingular in $d=1$ and numerically small. The agreement between the numerical simulations and the analytical method is good, indicating that the mode coupling approach seems to capture essential properties of the strong coupling fixed point behavior.

The heuristic and preliminary character of the scaling function given here does not allow a detailed comparison. We note, however, that since $A\propto\lambda(\Delta/\nu)^{1/2}$ in the Burgers-KPZ case the dimensionless argument in the scaling function $f(w)$ is $\lambda[(\Delta/\nu)^{1/2}t/x^{3/2}]$. This is in complete agreement with the driven lattice gas DRG analysis in [17] and with the general arguments advanced in the mode coupling analysis in [54,55].

We also note the curious fact that the spectral form (11.6) for a gapless dispersion with exponent β bears resemblance to the form of the probability distribution for a one-dimensional Lévy flight with index $\mu=\beta$ [100]. The case $\mu=\beta=2$ corresponds to ordinary Brownian walk, whereas $\mu=\beta=3/2<2$ is equivalent to super diffusion.

The present analysis of the scaling properties based on the spectral form (11.6) originates from a weak noise saddle point approximation to the path integral and as such only holds for $\Delta\rightarrow0$. However, within the general assumptions underlying the application of scaling theory and the notion of universality classes, we expect the exponents and scaling function to be universal characteristics of the system and thus *independent* of the noise strength Δ. This property can, however, be reconciled within the present many-body approach if we assume that an enhancement of the noise strength, that is a stronger drive of the system, only leads to a *dressing* of the quasiparticle spectrum, i.e., a change in the stiffness constant, and not to a change in the exponent β. In the ''quantum mechanical'' language this corresponds to the assumption that the WKB approximation also holds in the strong ''quantum regime'' as far as the exponent of the quasiparticle dispersion law is concerned.

We conclude this section with a few speculative remarks concerning the ''breakdown of hydrodynamics.'' The noisy Burgers equation is basically a nonlinear conserved hydrodynamical equation derived by combining the conservation law $\partial u/\partial t+\nabla u=0$ with a constitutive equation for the current, $j=-\nu\nabla u-(\lambda/2)u^2-\eta$, with transport coefficients ν (the damping) and λ (the mode coupling). The expression for the deterministic part of j is thus based on a gradient expansion to lowest order and the simplest quadratic nonlinearity in u. The issue is in which way the mode coupling term affects the hydrodynamical properties. In this context ''breakdown of hydrodynamics'' usually refers to the situa-

tion where the underlying regularity structure of the gradient expansion, i.e., in wave number space regularity in an expansion in k, breaks down.

In the present many-body formulation, entailing the spectral form (11.6), in frequency space we obtain

$$\langle uu \rangle (k,\omega) \sim \text{Re} \, \frac{1}{\omega - \Omega_k}. \qquad (11.12)$$

In the linear EW case for $\omega_k^0 = -i\nu k^2$ we recover the diffusive form (2.8), corresponding to a diffusive pole $\omega_k^0 = -i\nu k^2$ in the complex ω plane. However, in the nonlinear mode coupling case for $\lambda \neq 0$, $\Omega_k \propto |k|^{3/2}$ and $\langle uu \rangle (k,\omega)$ develops a branch cut structure, corresponding to a nonanalytic wave number dependence in the current, i.e., a breakdown of hydrodynamics.

XII. DISCUSSION AND CONCLUSION

In the present paper we have advanced an approach to the growth morphology and scaling behavior of the noisy Burgers equation in one dimension. Using the Martin-Siggia-Rose (MSR) technique in a canonical form we have demonstrated that the physics of the strong coupling fixed point is associated with an essential singularity in the noise strength and can be accessed by appropriate theoretical soliton techniques.

The canonical representation of the MSR functional integral in terms of a Feynman phase space path integral with a complex Hamiltonian identifies the noise strength as the relevant small nonperturbative parameter and allows for *a principle of least action*. In the asymptotic weak noise limit the leading contributions to the path integral are given by a dilute gas of solitons with superposed linear diffusive modes. The canonical variables are the local slope of the interface and an associated ''conjugate'' noise field, characteristic of the MSR formalism. In terms of the local slope the soliton and diffusive mode picture provide a many-body description of a growing interface governed by the noisy Burgers equation. The noise-induced slope fluctuations are here represented by the various paths or configurations contributing to the path integral.

The canonical formulation of the path integral and the associated principle of least action also allow us to associate energy, momentum, and action with a given soliton configuration or growth morphology. This gives rise to *a dynamical selection criterion* similar to the role of the Boltzmann factor $\exp(-E/T)$ in equilibrium statistical mechanics that associates an energy E with a given configuration contributing to the partition function; in the dynamical case the action S provides the weight function for the dynamical configuration. In more detail, in the dynamical case, by ''rotating'' the noise variable, $\varphi \rightarrow -i\varphi$, we obtain the partition function

$$Z_{\text{dyn}} \propto \int \prod_{xt} du \, d\varphi \, \exp\left[-\frac{\nu}{\Delta} \widetilde{S} \right], \qquad (12.1)$$

$$\widetilde{S} = \int dx \, dt \left[u \frac{\partial u}{\partial t} - \widetilde{\mathcal{H}}(u,\varphi) \right], \qquad (12.2)$$

$$\widetilde{\mathcal{H}} = -\frac{\nu}{2}[(\nabla u)^2 - (\nabla \varphi)^2] + \frac{\lambda}{2} u^2 \nabla \varphi, \qquad (12.3)$$

whereas in the equilibrium case we have the general form in one dimension,

$$Z_{eq} \propto \int \prod_x dp \, dq \, \exp\left[-\frac{1}{T} H \right], \qquad (12.4)$$

$$H = \int dx \, \mathcal{H}(p,q), \qquad (12.5)$$

where $\mathcal{H}$ is the Hamiltonian density. By comparison we note that the noise strength Δ plays the role of a ''temperature'' in the dynamical case. We also observe that Z_{dyn} for the dynamical 1D problem, treating time as an additional coordinate, i.e., $t \rightarrow y$, is equivalent to a 2D equilibrium partition function with the Hamiltonian

$$H = \int dx \, dy \left[u \nabla_y \varphi + \frac{\nu}{2}[(\nabla_x u)^2 - (\nabla_x \varphi)^2] - \frac{\lambda}{2} u^2 \nabla_x \varphi \right] \qquad (12.6)$$

and temperature Δ/ν.

In addition to providing a physical many-body picture of the morphology of a growing interface in terms of soliton modes accounting for the growth aspects and diffusive modes corresponding to the relaxational aspects, the present approach also gives insight into the scaling properties. The perspective here is not a ''coarse graining'' procedure, replacing the original description by a scaling description with ensuing dynamical renormalization group (DRG) equations, but rather a focus on the gapless elementary excitations or quasiparticles of the many-body theory.

The case of simple scaling characterized by a roughness exponent, a dynamical exponent, and a scaling function, corresponding to a simple fixed point structure in the DRG analysis, is here represented by a single quasiparticle mode exhausting the bottom of the spectrum with a gapless dispersion law. The dynamic exponent is given by the exponent in the quasiparticle dispersion law, whereas the roughness exponent follows from a regularity property of the form factor in a spectral representation of the slope correlations, with the scaling function being given by the spectral form itself.

Our analysis shows that the nonequilibrium growth dynamics in one dimension is controlled by solitons or dynamic domain wall. In this respect there is a parallel between the present kinetic growth problem and other well-studied low dimensional equilibrium problems also controlled by localized excitations such as the one-dimensional Ising model with domain wall excitations or the two-dimensional XY model characterized by vortex excitations [36]. The present approach is conducted in one dimension and assumes a spatially short range correlated conserved noise in order to implement the shift transformation leading to the canonical formulation and the separation of the Hamiltonian in a har

monic part and an interacting part. In higher dimensions the Burgers equation becomes a vector equation with a nonlinear term $\lambda(\vec{u}\cdot\vec{\nabla})\vec{u}$ and we obtain a more complicated Hamiltonian governing the dynamics.

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